

One-point functions of non-SUSY operators at arbitrary genus in a matrix model for type IIA superstrings

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Abstract

In the previous paper, the authors pointed out correspondence between a supersymmetric double-well matrix model and two-dimensional type IIA superstring theory on a Ramond-Ramond background from the viewpoint of symmetry and spectrum. This was confirmed by agreement between planar correlation functions in the matrix model and tree-level amplitudes in the superstring theory. In order to investigate the correspondence further, in this paper we compute correlation functions to all order of genus expansion in the double scaling limit of the matrix model. One-point functions of operators protected by supersymmetry terminate at some finite order, whereas those of unprotected operators yield non-Borel summable series. The behavior of the latter is characteristic in string perturbation series, providing further evidence that the matrix model describes a string theory. Moreover, instanton corrections to the planar one-point functions are also computed, and universal logarithmic scaling behavior is found for non-supersymmetric operators.

1 Introduction

The fact that the Large Hadron Collider (LHC) was not able to observe any supersymmetric particles at the first run has made supersymmetry breaking predicted to occur at higher energy scale. In this situation, it is meaningful to examine a possibility that supersymmetry would be broken in string scale itself rather than lower energy scale. On the other hand, by looking back on important roles played by supersymmetry in string theory, we are tempted to expect a possibility of spontaneous supersymmetry breaking in string theory. In view of these, it is undoubtedly important to explore mechanisms of spontaneous supersymmetry breaking in lower-dimensional string theory, namely a toy model of the critical string theory.

In the previous work [1], we pointed out correspondence between a supersymmetric matrix model with the scalar potential of double-well type and two-dimensional type IIA superstring theory on a Ramond-Ramond background. The correspondence has been made based on symmetry and spectrum in both sides. In [2], it is explicitly checked by comparing planar correlation functions in the matrix model and those at the tree level in the type IIA superstring theory. Based on the correspondence, it is expected that the matrix model nonperturbatively realizes the IIA superstring theory by taking its double scaling limit and enables nonperturbative investigation for the superstring theory. In [3, 4], we found that supersymmetries of the matrix model are spontaneously broken due to nonperturbative effects and the breaking persists in the double scaling limit, which suggests spontaneous breaking of target-space supersymmetries of the IIA theory by its nonperturbative contribution.

Of course, it is better to make the correspondence firmer by collecting its further evidence, in particular by agreement of amplitudes beyond the planar or tree level in both sides. In this paper, we present the result of one-point functions in the matrix model to all order of genus expansion and their corrections by nonperturbative instanton configurations. The forthcoming paper [5] is devoted to the all-order result of two-point functions¹. In these two papers, we focus on the correlation functions among single-trace operators of a matrix ϕ :

$$\frac{1}{N} \text{tr} \phi^n \quad (n \in \mathbf{N}). \quad (1.1)$$

For odd n , the operator (accompanied with operator mixing) corresponds to a vertex operator of Ramond-Ramond two-form field strength in the IIA theory. This is not protected by supersymmetry, while the operator for even n is protected. We will see that the correlation functions exhibit totally different behavior depending on odd or even n .

The organization of this paper is as follows. We give a brief review of the supersymmetric double-well matrix model in the next section. Infinitely degenerate vacua appearing

¹It seems hard to accomplish direct calculation of multi-point amplitudes in the matrix model and worldsheet computation in the IIA superstrings at higher genus. One of smarter ways may be to show that the Schwinger-Dyson equations in both sides coincide. Then agreement of arbitrary correlation functions at each order in perturbation theory automatically follows by matching boundary conditions of these equations. See e.g.[6], [7] for such attempts in the context of the IIB matrix model [8], and the Dijkgraaf-Vafa theory [9], respectively.

in the large- N limit are labeled by filling fractions, and we define the partition function with definite filling fraction for finite N . In section 3, we develop a general formalism for correlation functions among (1.1) at arbitrary genus for a fixed filling fraction. They are obtained from correlation functions of the resolvent operators

$$R_2(z) \equiv \frac{1}{N} \text{tr} \frac{1}{z - \phi^2}. \quad (1.2)$$

By the Nicolai mapping, the latter is expressed as correlation functions among the resolvent in the Gaussian matrix model at any genus ². According to this result, we explicitly compute the one-point functions in section 4. We then confirm the double scaling limit we proposed before indeed works. For even n , the genus expansion of the one-point functions terminates at some finite order and each term does not exhibit any singular behavior in the double scaling limit, which is plausible from the viewpoint of supersymmetry-protected operators or observables in the $c = -2$ topological gravity. On the other hand, for odd n , the expansion yields a non-Borel summable series, and each term is singular and exhibits universal behavior in the double scaling limit. The series grows as $(2h)!$ for large genus h , which is characteristic behavior in string perturbation series. In section 5, we consider the planar one-point functions in the presence of instantons in the matrix model, namely the leading order of perturbation around nonperturbative objects. Subtracting singular nonuniversal parts is necessary in this case differently from the perturbation theory without instanton. We see that the subtraction by operator mixing considered in computing cylinder amplitudes in [1] also works here. Validity of the double scaling limit is again confirmed. In section 6, we turn to correlation functions evaluated by the total partition function in the full sector, namely without specifying the filling fraction. By introducing a regularization parameter, we show that possible divergence in the full sector can be consistently absorbed by a kind of “wave-function renormalization” of odd-power operators ((1.1) with odd n), and thus well-defined correlation functions can be defined. Section 7 is devoted to discussions. In appendix A, we solve a recursion relation for coefficients in genus expansion of the resolvent in the Gaussian matrix model. In appendix B, we compute the one-point functions of even-power operators ((1.1) with even n) in a more general setting than the text. In appendix C, the instanton effect in section 5 is reproduced from the viewpoint of distortion of the eigenvalue distribution by the instantons.

2 Review of the supersymmetric matrix model

We consider a supersymmetric matrix model defined by the action:

$$S = N \text{tr} \left[\frac{1}{2} B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}(\phi\psi + \psi\phi) \right], \quad (2.1)$$

² As discussed in [1], the Nicolai mapping transforms the operators with even n to single-trace operators of integer power of matrices, which are observables in the $c = -2$ topological gravity. On the other hand, those with odd n become single-trace operators of half-integer power of matrices, which no longer belong to the observables in the topological gravity.

where B , ϕ are Grassmann even, and ψ , $\bar{\psi}$ are Grassmann odd $N \times N$ Hermitian matrices, respectively. By completing the square with respect to B , we find that the scalar potential for ϕ is of double-well type. The action S is invariant under supersymmetry transformations generated by Q and \bar{Q} :

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \quad (2.2)$$

and

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \quad (2.3)$$

which lead to the nilpotency: $Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$.

As shown in [10, 11], the planar limit of the matrix model for $\mu^2 \geq 2$ has infinitely degenerate supersymmetric vacua parametrized by filling fractions (ν_+, ν_-) , which represent configurations that $\nu_{\pm}N$ of the eigenvalues of ϕ are around the minimum $x = \pm|\mu|$ of the double-well potential $\frac{1}{2}(x^2 - \mu^2)^2$. On the other hand, that for $\mu^2 < 2$ has a vacuum which breaks the supersymmetry. The boundary $\mu^2 = 2$ is a critical point at which the third-order phase transition occurs. A simple large- N limit (planar limit) remains only the planar diagrams (tree amplitudes in the corresponding string theory). In fact, we have explicitly seen in [2] that the result of several types of correlation functions in the matrix model [1] reproduces the tree amplitudes in two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond background. As a limit yielding amplitudes beyond the planar ones, we consider the following double scaling limit [3] that approaches the critical point from the inside of the supersymmetric phase:

$$N \rightarrow \infty, \quad \mu^2 \rightarrow 2 + 0, \quad \text{with} \quad s = N^{\frac{2}{3}}(\mu^2 - 2) : \text{fixed}. \quad (2.4)$$

This limit of the matrix model is expected to provide nonperturbative formulation of the superstring theory with string coupling constant g_s proportional to $s^{-\frac{3}{2}}$. In [3, 4], instanton contribution to the free energy of the matrix model is found to have a factor $\exp\left(-\frac{C}{g_s}\right)$ with a constant C of $\mathcal{O}(1)$. This form is typical of solitonic objects in string theory (D-branes). Furthermore, the instantons cause spontaneous supersymmetry breaking in the matrix model, which implies violation of target-space supersymmetry by nonperturbative effects in the corresponding superstring theory.

In this paper we are interested in correlation functions at higher genera in the double scaling limit (2.4) in each sector with a fixed filling fraction. More precisely, in terms of the eigenvalues of ϕ , the partition function of (2.1) is given as [1, 3]

$$\begin{aligned} Z &\equiv (-1)^{N^2} \int d^{N^2} B d^{N^2} \phi \left(d^{N^2} \psi d^{N^2} \bar{\psi} \right) e^{-S} \\ &= \tilde{C}_N \int_{-\infty}^{\infty} \left(\prod_{i=1}^N 2\lambda_i d\lambda_i \right) \Delta(\lambda^2)^2 e^{-N \sum_{i=1}^N \frac{1}{2}(\lambda_i^2 - \mu^2)^2}, \end{aligned} \quad (2.5)$$

where the normalization of the measure is fixed by

$$\int d^{N^2} \phi e^{-N \text{tr}(\frac{1}{2}\phi^2)} = \int d^{N^2} B e^{-N \text{tr}(\frac{1}{2}B^2)} = 1 \quad (2.6)$$

and

$$(-1)^{N^2} \int \left(d^{N^2} \psi d^{N^2} \bar{\psi} \right) e^{-N \text{tr}(\bar{\psi} \psi)} = 1. \quad (2.7)$$

\tilde{C}_N is a constant dependent only on N : $\tilde{C}_N = (2\pi)^{-\frac{N}{2}} N^{\frac{N^2}{2}} \left(\prod_{k=0}^N k! \right)^{-1}$ [11], and $\Delta(x)$ stands for the Vandermonde determinant for eigenvalues x_i ($i = 1, \dots, N$): $\Delta(x) \equiv \prod_{i>j} (x_i - x_j)$. Namely, $\Delta(\lambda^2) = \prod_{i>j} (\lambda_i^2 - \lambda_j^2)$. By dividing the integration region of each λ_i according to the filling fraction, the total partition function can be expressed as a sum of each partition function with a fixed filling fraction:

$$Z = \sum_{\nu_- N=0}^N \frac{N!}{(\nu_+ N)! (\nu_- N)!} Z_{(\nu_+, \nu_-)},$$

$$Z_{(\nu_+, \nu_-)} \equiv \tilde{C}_N \int_0^\infty \left(\prod_{i=1}^{\nu_+ N} 2\lambda_i d\lambda_i \right) \int_{-\infty}^0 \left(\prod_{j=\nu_+ N+1}^N 2\lambda_j d\lambda_j \right) \Delta(\lambda^2)^2 e^{-N \sum_{m=1}^N \frac{1}{2} (\lambda_m^2 - \mu^2)^2}. \quad (2.8)$$

By changing the integration variables $\lambda_j \rightarrow -\lambda_j$ ($j = \nu_+ N + 1, \dots, N$), it is easy to find

$$Z_{(\nu_+, \nu_-)} = (-1)^{\nu_- N} Z_{(1,0)}, \quad (2.9)$$

and therefore the total partition function vanishes:³

$$Z = \sum_{\nu_- N=0}^N \frac{N!}{(\nu_+ N)! (\nu_- N)!} (-1)^{\nu_- N} Z_{(1,0)} = (1 + (-1))^N Z_{(1,0)} = 0. \quad (2.10)$$

We define the correlation function of K single-trace operators $\frac{1}{N} \text{tr} \mathcal{O}_a(\phi)$ ($a = 1, \dots, K$) in the (ν_+, ν_-) sector as

$$\left\langle \prod_{a=1}^K \frac{1}{N} \text{tr} \mathcal{O}_a(\phi) \right\rangle^{(\nu_+, \nu_-)} \equiv \frac{\tilde{C}_N}{Z_{(\nu_+, \nu_-)}} \int_0^\infty \left(\prod_{i=1}^{\nu_+ N} 2\lambda_i d\lambda_i \right) \int_{-\infty}^0 \left(\prod_{j=\nu_+ N+1}^N 2\lambda_j d\lambda_j \right) \Delta(\lambda^2)^2$$

$$\times \left(\prod_{a=1}^K \frac{1}{N} \sum_{i=1}^N \mathcal{O}_a(\lambda_i) \right) e^{-N \sum_{m=1}^N \frac{1}{2} (\lambda_m^2 - \mu^2)^2}, \quad (2.11)$$

and express its connected part by the $1/N$ -expansion:

$$\left\langle \prod_{a=1}^K \frac{1}{N} \text{tr} \mathcal{O}_a(\phi) \right\rangle_C^{(\nu_+, \nu_-)} = \sum_{h=0}^{\infty} \frac{1}{N^{2h+2K-2}} \left\langle \prod_{a=1}^K \frac{1}{N} \text{tr} \mathcal{O}_a(\phi) \right\rangle_{C,h}^{(\nu_+, \nu_-)}. \quad (2.12)$$

³ The consequence directly follows from the fact that the integrand of the total partition function (2.5) is odd under the sign flip of an arbitrary eigenvalue.

$\langle \cdot \rangle_{C,h}^{(\nu_+, \nu_-)}$ denotes the connected correlation function on a handle- h random surface with the N -dependence factored out; i.e., the quantity of $\mathcal{O}(N^0)$. In this paper, we focus on the case that $\mathcal{O}_a(\phi)$ are polynomials of ϕ . Operators $\frac{1}{N} \text{tr} B^k$ or equivalently (linear combinations of) $\frac{1}{N} \text{tr} \phi^{2k}$ ($k \in \mathbf{N} \cup \{0\}$) are invariant under the supersymmetries (2.2) and (2.3). Correlation functions among them do not exhibit any nonanalytic behavior as $s \rightarrow 0$ at the planar level ($h = 0$) [1], which is characteristic of protection by supersymmetry. On the other hand, operators of odd powers: $\frac{1}{N} \text{tr} \phi^{2k+1}$ ($k \in \mathbf{N} \cup \{0\}$) are not invariant under either of Q or \bar{Q} , and show nontrivial critical behavior as power of $\ln s$ at the planar level [1]:

$$\left\langle \prod_{a=1}^K \Phi_{2k_a+1} \right\rangle_{C,0}^{(\nu_+, \nu_-)} \Big|_{\text{sing.}} = (\nu_+ - \nu_-)^K (\text{const.}) s^{2-\gamma+\sum_{a=1}^K (k_a-1)} (\ln s)^K + (\text{less singular at } s=0) \quad (2.13)$$

with the string susceptibility exponent $\gamma = -1$. Here, “ $|_{\text{sing.}}$ ” stands for ignoring regular functions of s at $s = 0$, and the factor “(const.)” contains a certain power of N . Φ_{2k+1} is essentially $\frac{1}{N} \text{tr} \phi^{2k+1}$ with operator mixing:

$$\Phi_{2k+1} = \frac{1}{N} \text{tr} \phi^{2k+1} + (\text{mixing}) \quad (k \in \mathbf{N} \cup \{0\}), \quad (2.14)$$

where “mixing” represents a sum of even powers of ϕ lower than the degree $2k+1$. For instance, an explicit form is given in (5.29). These are introduced in order to remove nonuniversal singular terms in the double scaling limit. (2.13) are expected to correspond to correlation functions of the Ramond-Ramond fields in the two-dimensional type IIA superstring theory [1]. By computing scattering amplitudes in the superstring side and comparing the result with (2.13), the expectation has been confirmed for one- and two-point functions ($K = 1, 2$) with arbitrary odd powers [2]. Thus it is desirable to go beyond the tree level and get an expression as in (2.13) at higher genus, which is our main motivation.

3 Correlation functions at arbitrary genus

In order to compute one-point functions for a function $\mathcal{O}(\phi)$ of ϕ in the filling fraction $(1, 0)$ at arbitrary genus h : $\langle \frac{1}{N} \text{tr} \mathcal{O}(\phi) \rangle_h^{(1,0)}$, we consider the ϕ^2 -resolvent

$$\langle R_2(z^2) \rangle_h^{(1,0)} = \left\langle \frac{1}{N} \text{tr} \frac{1}{z^2 - \phi^2} \right\rangle_h^{(1,0)} \quad (3.1)$$

rather than the standard resolvent $\left\langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \right\rangle_h^{(1,0)}$. The former is protected by the supersymmetry and is expected to have a simpler form compared with the latter. In terms of

the eigenvalues, $R_2(z^2)$ is

$$\frac{1}{N} \sum_{i=1}^N \frac{1}{z^2 - \lambda_i^2} = \frac{1}{N} \frac{1}{2z} \sum_{i=1}^N \left(\frac{1}{z - \lambda_i} + \frac{1}{z + \lambda_i} \right) \quad (3.2)$$

and $\frac{1}{z - \lambda_i}$ ($\frac{1}{z + \lambda_i}$) has poles only on the positive (negative) real axis in the $(1, 0)$ filling fraction. At each order in the $1/N$ -expansion, the poles accumulate to develop a cut $[a, b]$ ($[-b, -a]$), where $a = \sqrt{\mu^2 - 2}$ and $b = \sqrt{\mu^2 + 2}$ [1]. Thus the one-point function is given by the contour integral of (3.1):⁴

$$\left\langle \frac{1}{N} \text{tr } \mathcal{O}(\phi) \right\rangle_h^{(1,0)} = \oint_{[a,b], z} 2z \cdot \mathcal{O}(z) \langle R_2(z^2) \rangle_h^{(1,0)}, \quad (3.3)$$

where $\oint_D, z \equiv \oint_D \frac{dz}{2\pi i}$ denotes the z -integral along the contour encircling only the region D counterclockwise⁵. The case of $D = [a, b]$ is depicted in Fig. 1. In the case of $\mathcal{O}(\phi) = \phi^n$ as in (1.1), the one-point function for a general filling fraction $\langle \frac{1}{N} \text{tr } \phi^n \rangle_h^{(\nu_+, \nu_-)}$ is obtained by simply multiplying the factor $(\nu_+ - \nu_-)^\sharp$ to (3.3), where $\sharp = 0$ and 1 for even and odd n , respectively [1].

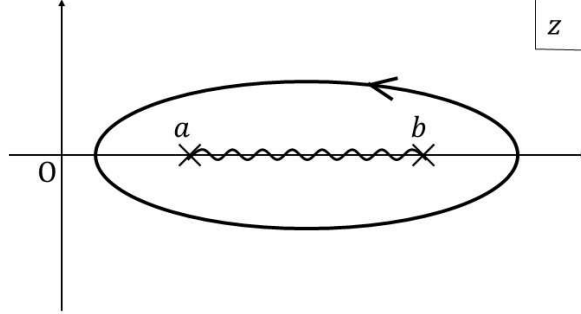


Fig. 1: Integration contour on the complex z -plane.

It is easy to extend this argument to multi-point functions. The K -point connected correlation function among $\frac{1}{N} \text{tr } \mathcal{O}_a(\phi)$ ($a = 1, \dots, K$) is obtained from the K -point function of the ϕ^2 -resolvent as

$$\left\langle \prod_{\ell=1}^K \frac{1}{N} \text{tr } \mathcal{O}_\ell(\phi) \right\rangle_{C, h}^{(1,0)} = \prod_{\ell=1}^K \oint_{[a,b], z_\ell} 2z_\ell \cdot \mathcal{O}_\ell(z_\ell) \left\langle \prod_{a=1}^K R_2(z_a^2) \right\rangle_{C, h}^{(1,0)}. \quad (3.4)$$

⁴Here we implicitly assume that $\mathcal{O}(z)$ itself does not have singularity on the cut $[a, b]$.

⁵ In this paper, we treat cases where D is an interval or a point. When D is their union, the contour should be understood as $\oint_{D_1 \cup D_2, z} = \oint_{D_1, z} + \oint_{D_2, z}$.

Let us next note that the ϕ^2 -resolvent is mapped to the resolvent in a Gaussian matrix model. In fact, the Nicolai mapping $x_i = \mu^2 - \lambda_i^2$ ($i = 1, \dots, N$) recasts the partition function $Z_{(1,0)}$ and the correlation function (2.11) in the $(1, 0)$ sector as

$$Z_{(1,0)} = \tilde{C}_N \int_{-\infty}^{\mu^2} \left(\prod_{i=1}^N dx_i \right) \Delta(x)^2 e^{-N \sum_{i=1}^N \frac{1}{2} x_i^2} \equiv Z^{(G')} \quad (3.5)$$

and

$$\left\langle \prod_{a=1}^K \frac{1}{N} \text{tr} \mathcal{O}_a(\phi) \right\rangle^{(1,0)} = \frac{\tilde{C}_N}{Z^{(G')}} \int_{-\infty}^{\mu^2} \left(\prod_{i=1}^N dx_i \right) \Delta(x)^2 \left(\prod_{a=1}^K \frac{1}{N} \sum_{i=1}^N \mathcal{O}_a(\sqrt{\mu^2 - x_i}) \right) \times e^{-N \sum_{i=1}^N \frac{1}{2} x_i^2}, \quad (3.6)$$

respectively. The integrals of the eigenvalues x_i are not over the entire real line, but are bounded from above by μ^2 . We put the superscript ‘ (G') ’ on quantities in such a Gaussian matrix model. The difference from the standard one whose eigenvalues are integrated over the whole real axis is nonperturbative in $1/N$ and negligible in the genus expansion [3]. As pointed out in [1], supersymmetric operators $\frac{1}{N} \text{tr} \phi^{2k}$ are mapped onto observables in the $c = -2$ topological gravity (the standard Gaussian matrix model); i.e., polynomials in x_i , while non-supersymmetric operators $\frac{1}{N} \text{tr} \phi^{2k+1}$ are not due to the branch cut singularity of the square root. In particular, $R_2(z^2)$ becomes

$$\frac{1}{N} \sum_{i=1}^N \frac{1}{z^2 - \mu^2 + x_i} = -R_M(\mu^2 - z^2) \quad (3.7)$$

in terms of eigenvalues, where $R_M(x) \equiv \frac{1}{N} \text{tr} \frac{1}{x-M}$ is the resolvent in the Gaussian matrix model whose matrix variable M is an $N \times N$ Hermitian matrix and its eigenvalues are x_i ($i = 1, \dots, N$). Now the problem is reduced to higher-genus correlation functions in the Gaussian matrix model:

$$\left\langle \frac{1}{N} \text{tr} \mathcal{O}(\phi) \right\rangle_h^{(1,0)} = - \oint_{[a,b], z} 2z \cdot \mathcal{O}(z) \langle R_M(\mu^2 - z^2) \rangle_h^{(G)} \quad (3.8)$$

and

$$\left\langle \prod_{a=1}^K \frac{1}{N} \text{tr} \mathcal{O}_a(\phi) \right\rangle_{C,h}^{(1,0)} = \prod_{\ell=1}^K \left(- \oint_{[a,b], z_\ell} 2z_\ell \cdot \mathcal{O}_\ell(z_\ell) \right) \left\langle \prod_{a=1}^K R_M(\mu^2 - z_a^2) \right\rangle_{C,h}^{(G)}. \quad (3.9)$$

The superscript ‘ (G) ’ (not ‘ (G') ’) on the r.h.s. indicates the use of the standard Gaussian matrix model, which is allowed in the genus expansion from the reason mentioned above. In particular,

$$\left\langle \prod_{a=1}^K \frac{1}{N} \text{tr} \phi^{n_a} \right\rangle_{C,h}^{(1,0)} = \prod_{\ell=1}^K \left(- \oint_{[a,b], z_\ell} 2z_\ell \cdot z_\ell^{n_\ell} \right) \left\langle \prod_{a=1}^K R_M(\mu^2 - z_a^2) \right\rangle_{C,h}^{(G)}. \quad (3.10)$$

It is interesting to point out that correlation functions involving not only even-power operators but also odd-power ones are obtained from the supersymmetric ϕ^2 -resolvent as in (3.10). It suggests that infinitely many supersymmetric local operators ($\frac{1}{N}\text{tr } \phi^{2k}$ ($k \in \mathbf{N}$)) that are equivalent to the resolvent operator (1.2) can also carry information for non-supersymmetric operators.

4 One-point functions

In this section, we give explicit formulas of the one-point functions of the operators (1.1) at arbitrary genus from (3.10).

According to the literature in the random matrix theory, e.g. [12], the $1/N$ -expansion of the resolvent in the Gaussian matrix model is explicitly given as

$$\langle R_M(x) \rangle^{(G)} = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} \eta_h(x) \quad (4.1)$$

with

$$\eta_0(x) = \frac{1}{2}x - \frac{1}{2}(x^2 - 4)^{\frac{1}{2}}, \quad (4.2)$$

$$\eta_j(x) = \sum_{r=2j}^{3j-1} C_{j,r} (x^2 - 4)^{-r-\frac{1}{2}} \quad (j \in \mathbf{N}). \quad (4.3)$$

The coefficients $C_{j,r}$ satisfy a recursion relation

$$C_{j+1,r} = \frac{(2r-3)(2r-1)}{r+1} ((r-1)C_{j,r-2} + (4r-10)C_{j,r-3}) \quad (4.4)$$

for $2j+2 \leq r \leq 3j+2$ with conditions

$$C_{j,2j-1} = C_{j,3j} = 0, \quad C_{1,2} = 1. \quad (4.5)$$

The use of this result in (3.10) with $x = \mu^2 - z_1^2$ leads to

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr } \phi^n \right\rangle_0^{(1,0)} &= -\frac{1}{2} I_{1,n}, \\ \left\langle \frac{1}{N} \text{tr } \phi^n \right\rangle_j^{(1,0)} &= \sum_{r=2j}^{3j-1} C_{j,r} I_{-2r-1,n} \quad (j \in \mathbf{N}), \end{aligned} \quad (4.6)$$

where

$$I_{m,n} \equiv \oint_{[-2,2],x} (x^2 - 4)^{\frac{m}{2}} (\mu^2 - x)^{\frac{n}{2}}. \quad (4.7)$$

4.1 Computation of universal contribution

We evaluate universal contribution to the one-point functions (4.6) in the double scaling limit (2.4). When n is an even integer, $I_{m,n}$ becomes a polynomial of μ^2 and the one-point functions do not exhibit any nonanalytic behavior as $\mu^2 \rightarrow 2$. Therefore, we focus on the case where n is odd ($n = 2k + 1$, $k \in \mathbf{N} \cup \{0\}$) in this subsection. Then the mixing term in (2.14) solely gives uninteresting analytic contribution to the one-point functions, and $\langle \Phi_{2k+1} \rangle_h^{(1,0)}$ is identical with $\langle \frac{1}{N} \text{tr} \phi^{2k+1} \rangle_h^{(1,0)}$ regarding their universal (nonanalytic) contribution.

Let us change variables so that they will magnify the vicinity of the critical point in the double scaling limit as $\mu^2 = 2 + N^{-\frac{2}{3}}s$, $x = 2 - N^{-\frac{2}{3}}\xi$. Then

$$I_{m,2k+1} = - \left(N^{-\frac{2}{3}} \right)^{\frac{m+3}{2}+k} (-2i)^m \oint_{[0,\infty),\xi} \xi^{\frac{m}{2}} (s + \xi)^{k+\frac{1}{2}} \left(1 + \mathcal{O}(N^{-\frac{2}{3}}) \right) \quad (4.8)$$

with $m = 1, -1, -3, \dots$ and $k = 0, 1, 2, \dots$. Since $x = 2$ and $x = -2$ are mapped to $\xi = 0$ and $\xi = 4N^{\frac{2}{3}}$ respectively, the integration contour becomes to surround the positive real axis in the double scaling limit as in Fig. 2.

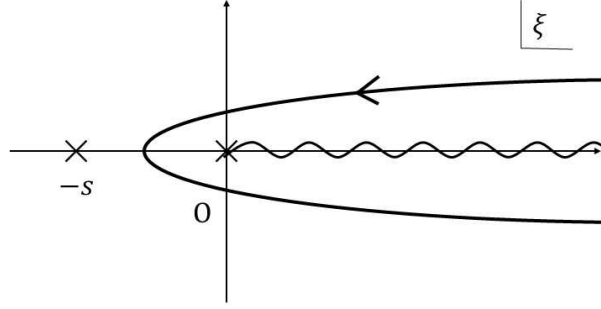


Fig. 2: Integration contour on the complex ξ -plane.

The integration around the critical point $\xi = 0$ provides the universal contribution, while possible divergence from the integration around $\xi = \infty$ will become cut-off dependent (N -dependent) analytic terms of s (nonuniversal contribution). The latter is discarded by taking s -derivatives $(k + 3)$ times:

$$\frac{\partial^{k+3}}{\partial s^{k+3}} I_{m,2k+1} = - \left(N^{-\frac{2}{3}} \right)^{\frac{m+3}{2}+k} (-2i)^m \frac{\Gamma(k + \frac{3}{2})}{\Gamma(-\frac{3}{2})} \oint_{[0,\infty),\xi} \xi^{\frac{m}{2}} (s + \xi)^{-\frac{5}{2}} \left(1 + \mathcal{O}(N^{-\frac{2}{3}}) \right). \quad (4.9)$$

By rescaling ξ as $\xi \rightarrow s\xi$, the integral is expressed by the Beta function as ⁶

$$\oint_{[0,\infty),\xi} \xi^{\frac{m}{2}} (1+\xi)^{-\frac{5}{2}} = \frac{e^{i\pi m} - 1}{2\pi i} B\left(\frac{m}{2} + 1, \frac{3-m}{2}\right). \quad (4.10)$$

Thus we have

$$\frac{\partial^{k+3}}{\partial s^{k+3}} I_{m,2k+1} = \left(N^{-\frac{2}{3}}\right)^{\frac{m+3}{2}+k} (-i)^{m+1} \frac{2^m}{\pi^2} \Gamma\left(k + \frac{3}{2}\right) \Gamma\left(\frac{m}{2} + 1\right) \Gamma\left(\frac{3-m}{2}\right) s^{-\frac{3-m}{2}} \quad (4.11)$$

up to subleading contribution. Integrating $(k+3)$ times with respect to s leads to

$$I_{m,2k+1} = -\left(N^{-\frac{2}{3}}\right)^{\frac{m+3}{2}+k} \frac{2^m}{\pi^2} \frac{\Gamma\left(\frac{m}{2} + 1\right) \Gamma\left(k + \frac{3}{2}\right)}{\Gamma\left(\frac{m+5}{2} + k\right)} s^{\frac{m+3}{2}+k} \ln s + (\text{less singular}) \quad (4.12)$$

for $\frac{m+3}{2} + k \geq 0$, and

$$I_{m,2k+1} = \left(N^{-\frac{2}{3}}\right)^{\frac{m+3}{2}+k} (-1)^{\frac{m+3}{2}+k} \frac{2^m}{\pi^2} \Gamma\left(\frac{m}{2} + 1\right) \Gamma\left(k + \frac{3}{2}\right) \Gamma\left(-\frac{m+3}{2} - k\right) s^{\frac{m+3}{2}+k} + (\text{less singular}) \quad (4.13)$$

for $\frac{m+3}{2} + k < 0$. Here, “(less singular)” stands for less singular terms at $s = 0$ compared with the first term. More precisely, they are polynomials in $\mu^2 = 2 + N^{-\frac{2}{3}}s$ (cutoff dependent nonuniversal parts) or subleading terms in the double scaling limit (2.4). Since from the definition (4.7), $I_{m,n}$ depends on s and N only through the combination $N^{-\frac{2}{3}}s$, the r.h.s. of (4.12) should appear as

$$\left(N^{-\frac{2}{3}}s\right)^{\frac{m+3}{2}+k} \ln\left(N^{-\frac{2}{3}}s\right) = \left(N^{-\frac{2}{3}}s\right)^{\frac{m+3}{2}+k} \ln s - \frac{2}{3} \left(N^{-\frac{2}{3}}s\right)^{\frac{m+3}{2}+k} \ln N \quad (4.14)$$

up to the overall factor independent of s and N . Note that although the last term is larger than the first term in the double scaling limit, it belongs to the less singular terms around $s = 0$.

From (4.12) and (4.13), we see that the universal part of $I_{m,n}$ is more dominant for smaller m with k fixed. Hence in the sum in (4.6), $r = 3j - 1$ gives the most dominant contribution:

$$\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle_j^{(1,0)} = C_{j,3j-1} I_{-6j+1,2k+1} \quad (4.15)$$

⁶ Note that the expression of the Beta function $B(p, q) = \frac{2\pi i}{e^{i2\pi p} - 1} \oint_{[0,\infty),y} \frac{y^{p-1}}{(1+y)^{p+q}}$ is valid for a larger region $p \notin \mathbf{Z}$ and $\text{Re } q > 0$ compared with $\text{Re } p > 0$ and $\text{Re } q > 0$ where another integral representation $B(p, q) = \int_0^\infty dy \frac{y^{p-1}}{(1+y)^{p+q}}$ is available. Furthermore, the integral $\oint_{[0,\infty),y} \frac{y^{p-1}}{(1+y)^{p+q}}$ itself is well-defined for $p \in \mathbf{C}$ and $\text{Re } q > 0$. The combination $(e^{i2\pi p} - 1)\Gamma(p)$ has no singularity except for $p = \infty$.

in the double scaling limit. As in appendix A, the coefficient $C_{j,3j-1}$ can be solved in a simple form (A.4). It can be rewritten as

$$C_{j,3j-1} = \frac{1}{4\sqrt{\pi}} \left(\frac{16}{3}\right)^j \frac{\Gamma(3j - \frac{1}{2})}{j!}. \quad (4.16)$$

We thus find the relevant contribution to the one-point function (4.6) in the double scaling limit as

$$\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle_h^{(1,0)} \Big|_{\text{univ.}} = (N^{-\frac{2}{3}})^{k+2-3h} \frac{1}{2\pi^{\frac{3}{2}}} \frac{1}{h!} \left(-\frac{1}{12}\right)^h \frac{\Gamma(k + \frac{3}{2})}{\Gamma(k + 3 - 3h)} s^{k+2-3h} \ln s \quad (4.17)$$

for $0 \leq h \leq \frac{k+2}{3}$, and

$$\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle_h^{(1,0)} \Big|_{\text{univ.}} = (N^{-\frac{2}{3}})^{k+2-3h} \frac{(-1)^{k+1}}{2\pi^{\frac{3}{2}}} \frac{1}{h!} \left(\frac{1}{12}\right)^h \Gamma\left(k + \frac{3}{2}\right) \Gamma(3h - k - 2) s^{k+2-3h} \quad (4.18)$$

for $h > \frac{k+2}{3}$. The symbol “ $|_{\text{univ.}}$ ” means the most dominant nonanalytic term at $s = 0$ (the universal part) taken ⁷. In other words, recalling (2.12), we obtain the genus expansion of the universal part of the one-point function as

$$\begin{aligned} & \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle_h^{(1,0)} \Big|_{\text{univ., pert.}} \\ &= N^{-\frac{2}{3}(k+2)} \frac{\Gamma(k + \frac{3}{2})}{2\pi^{\frac{3}{2}}} \left[\sum_{h=0}^{\lfloor \frac{1}{3}(k+2) \rfloor} \frac{1}{h!} \left(-\frac{1}{12}\right)^h \frac{1}{\Gamma(k + 3 - 3h)} s^{k+2-3h} \ln s \right. \\ & \quad \left. + (-1)^{k+1} \sum_{h=\lfloor \frac{1}{3}(k+2) \rfloor + 1}^{\infty} \frac{1}{h!} \left(\frac{1}{12}\right)^h \Gamma(3h - k - 2) s^{k+2-3h} \right]. \end{aligned} \quad (4.19)$$

Here, the subscript “pert.” indicates the genus expansion (the expansion with respect to $s^{-3} \propto N^{-2} \propto g_s^2$) that corresponds to perturbative expansion in string theory ⁸. $\lfloor \frac{1}{3}(k+2) \rfloor$ denotes the greatest integer not exceeding $\frac{1}{3}(k+2)$. The result (4.19) explicitly shows that the double scaling limit (2.4) keeps contribution of each order in the genus expansion finite. The overall factor $N^{-\frac{2}{3}(k+2)}$ can be absorbed in the “wave function renormalization” of the operator $\frac{1}{N} \text{tr} \phi^{2k+1}$.

The logarithmic singularity appears only at lower genera. This reminds us of the case of the bosonic $c = 1$ noncritical string theory on two-dimensional target space. There, the free energy has the logarithmic behavior only at genus zero and one [13, 14].

⁷The $h = 0$ case in (4.17) reproduces the result given in [1] with $\omega = N^{-\frac{2}{3}} \frac{s}{4}$.

⁸ Here we do not take into account nonperturbative contribution from the boundary of the eigenvalue integration as mentioned in section 3.

We can read large-order behavior from (4.19). In fact, relevant higher-genus contribution is given in the second line on the r.h.s.. This is a positive term series growing as $(2h)!$:

$$(2h)! \left(\frac{4}{3} s^{\frac{3}{2}} \right)^{-2h} \quad (4.20)$$

for sufficiently large genus h . It is in contrast to the lower-genus contribution given in the first line that is an alternating series. The behavior (4.20) is a characteristic feature of string perturbation series, and gives a further support that the matrix model describes a string theory in the double scaling limit [15]. As discussed in [1], the matrix model with two supersymmetries (2.2) and (2.3) can be regarded as two-dimensional type IIA superstring theory with the corresponding target-space supersymmetries. In the matrix model, the supersymmetries are preserved to all order in the $1/N$ -expansion, while they are spontaneously broken due to nonperturbative effects [3, 4]. This indicates that the supersymmetries in the IIA superstring theory are not broken to all order in perturbation theory (expansion in $s^{-3}(=g_s^2)$), but are violated nonperturbatively.

Furthermore, as in [15], from (4.20) we can also deduce a nonperturbative effect as $\exp\left(-\frac{4}{3}s^{\frac{3}{2}}\right) \left(= \exp\left(-\frac{4}{3}\frac{1}{g_s}\right)\right)$, which in fact coincides with one calculated in [3] as contribution from an isolated eigenvalue (one-instanton effect). Note that this kind of large-order behavior would not be observed in supersymmetric quantities. For example, the free energy $(-\ln Z_{(1,0)})$ has no perturbative contribution and its trans-series expansion starts with one-instanton effect [3, 4]. Another example is the one-point functions of the even-power operators $\frac{1}{N}\text{tr } \phi^{2\ell}$ or $\frac{1}{N}\text{tr } B^\ell$ ($\ell \in \mathbf{N}$) that will be considered in the next subsection and appendix B. Their genus expansions terminate at some finite order. This fact originates from huge cancellation due to the supersymmetry. Thus we recognize that in order to predict nonperturbative effect from the large-order behavior in perturbation theory, we have to consider non-supersymmetric operators in general to prevent cancellation in perturbative series. Here we have explicitly observed it for the odd-power operators (4.19).

4.2 Computation of full contribution

In this subsection, we compute the full contribution to $I_{m,n}$ for $n \in \mathbf{N}$ including nonuniversal parts. The evaluation of the nonuniversal parts is relevant to fix the mixing terms in (2.14) and to obtain two-point functions in the next paper [5].

We change the integration variable as $x = -2 + 4t$ in (4.7) and express $I_{m,n}$ in terms of the hypergeometric function:

$$\begin{aligned} I_{m,n} &= 4(4i)^m b^n \oint_{[0,1],t} t^{\frac{m}{2}} (1-t)^{\frac{m}{2}} \left(1 - \frac{4}{b^2}t\right)^{\frac{n}{2}} \\ &= \frac{(4i)^{m+1}}{2\pi} (1 - (-1)^m) b^n \frac{\Gamma\left(\frac{m}{2} + 1\right)^2}{\Gamma(m+2)} F\left(-\frac{n}{2}, \frac{m}{2} + 1, m+2; \frac{4}{b^2}\right). \end{aligned} \quad (4.21)$$

We have fixed the branch of $(x^2 - 4)^{\frac{m}{2}}$ as

$$(x^2 - 4)^{\frac{m}{2}} = (4e^{i\frac{\pi}{2}})^m t^{\frac{m}{2}} (1 - t)^{\frac{m}{2}}, \quad (4.22)$$

where $t^{\frac{m}{2}}(1 - t)^{\frac{m}{2}}$ has no phase factor for $t = \tau + i0$ and $0 < \tau < 1$. In the first line of (4.21), m must be an integer for the integrand to be single-valued along the contour surrounding the cut $[0, 1]$. The hypergeometric function itself in the last line of (4.21)

$$F\left(-\frac{n}{2}, \frac{m}{2} + 1, m + 2; \frac{4}{b^2}\right) = \sum_{p=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_p \left(\frac{m}{2} + 1\right)_p}{(m + 2)_p} \frac{1}{p!} \left(\frac{4}{b^2}\right)^p \quad (4.23)$$

with $(x)_p \equiv x(x+1)\cdots(x+p-1)$ and $(x)_0 \equiv 1$ is not well-defined for $m = -3, -5, -7, \dots$ because of $(m+2)_p$ in the denominator. However, together with the prefactor $1/\Gamma(m+2)$, it becomes nonsingular:

$$\frac{1}{\Gamma(m+2)} \frac{1}{(m+2)_p} = \frac{1}{\Gamma(m+2+p)}. \quad (4.24)$$

Let us consider the case $m = -2r - 1$ ($r \geq 2$) in (4.6). Then, noting that nonvanishing contribution in the sum of p starts with $2r$ due to (4.24), we shift the variable p as $p \rightarrow p + 2r$ and recast $I_{-2r-1, n}$ in terms of another hypergeometric function that is clearly nonsingular :

$$I_{-2r-1, n} = \frac{\left(-\frac{n}{2}\right)_{2r}}{(2r)!} b^{n-4r} F\left(2r - \frac{n}{2}, r + \frac{1}{2}, 2r + 1; \frac{4}{b^2}\right). \quad (4.25)$$

The expression itself is also correct for $r = 0, 1$. The $m = 1$ case in (4.21) is also well-defined and from (4.6) the disk amplitude becomes

$$\left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} = -\frac{1}{2} I_{1, n} = b^n F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{4}{b^2}\right). \quad (4.26)$$

This agrees with the result obtained previously (eq. (3.1) in [1]).

When n is even: $n = 2\ell$ ($\ell \in \mathbf{N}$), $I_{-2r-1, 2\ell}$ is not null for $2r \leq \ell$ due to the factor $\left(-\frac{n}{2}\right)_{2r} = (-\ell)_{2r}$ in (4.25). In this case, $I_{-2r-1, 2\ell}$ reduces a polynomial of $b^2 = 2 + \mu^2$ with the degree $(\ell - 2r)$:

$$I_{-2r-1, 2\ell} = \frac{(-\ell)_{2r}}{(2r)!} b^{2\ell-4r} \sum_{p=0}^{\ell-2r} \frac{(2r - \ell)_p \left(r + \frac{1}{2}\right)_p}{(2r + 1)_p} \frac{1}{p!} \left(\frac{4}{b^2}\right)^p. \quad (4.27)$$

Plugging this and the result for $C_{j, r}$ in appendix A into (4.6) presents the full contribution to the higher-genus one-point functions $\left\langle \frac{1}{N} \text{tr} \phi^{2\ell} \right\rangle_j$ ($j \in \mathbf{N}$) as polynomials of $b^2 = 2 + \mu^2$. Since $I_{-2r-1, 2\ell} = 0$ for $\ell \leq 2r - 1$ and $I_{-4j-1, 8j} = 1$ ($j \in \mathbf{N}$) by (4.27), it is easy to see

from (4.6)

$$\left\langle \frac{1}{N} \text{tr} \phi^{2\ell} \right\rangle_j^{(1,0)} = 0 \quad \text{for } \ell \leq 4j - 1, \quad (4.28)$$

$$\left\langle \frac{1}{N} \text{tr} \phi^{8j} \right\rangle_j^{(1,0)} = C_{j,2j} = \frac{(4j-1)!!}{2j+1}, \quad (4.29)$$

where we have used (A.6). In addition to the planar contribution

$$\left\langle \frac{1}{N} \text{tr} \phi^2 \right\rangle_0^{(1,0)} = \mu^2, \quad \left\langle \frac{1}{N} \text{tr} \phi^4 \right\rangle_0^{(1,0)} = 1 + \mu^4, \quad \left\langle \frac{1}{N} \text{tr} \phi^6 \right\rangle_0^{(1,0)} = 3\mu^2 + \mu^6, \dots, \quad (4.30)$$

the first few nonvanishing expressions at each genus of $h = 1, 2, 3$ are

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi^8 \right\rangle_1^{(1,0)} &= 1, & \left\langle \frac{1}{N} \text{tr} \phi^{10} \right\rangle_1^{(1,0)} &= 5\mu^2, & \left\langle \frac{1}{N} \text{tr} \phi^{12} \right\rangle_1^{(1,0)} &= 10 + 15\mu^4, \dots, \\ \left\langle \frac{1}{N} \text{tr} \phi^{16} \right\rangle_2^{(1,0)} &= 21, & \left\langle \frac{1}{N} \text{tr} \phi^{18} \right\rangle_2^{(1,0)} &= 189\mu^2, & \left\langle \frac{1}{N} \text{tr} \phi^{20} \right\rangle_2^{(1,0)} &= 483 + 945\mu^4, \dots, \\ \left\langle \frac{1}{N} \text{tr} \phi^{24} \right\rangle_3^{(1,0)} &= 1485, & \left\langle \frac{1}{N} \text{tr} \phi^{26} \right\rangle_3^{(1,0)} &= 19305\mu^2, & \left\langle \frac{1}{N} \text{tr} \phi^{28} \right\rangle_3^{(1,0)} &= 56628 + 135135\mu^4, \\ &\dots & & & & \end{aligned} \quad (4.31)$$

On the other hand, for odd n : $n = 2k + 1$ ($k \in \mathbf{N} \cup \{0\}$), the full contribution to $\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle_h$ ($h \in \mathbf{N} \cup \{0\}$) exhibits singular behavior in the double scaling limit (2.4) due to the argument of the hypergeometric functions $\frac{4}{b^2} = \left(1 + N^{-\frac{2}{3}} \frac{s}{4}\right)^{-1}$ approaching 1 from below⁹. From (4.26), we have

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi \right\rangle_0^{(1,0)} &= \frac{64}{15\pi} + N^{-\frac{2}{3}} \frac{4s}{3\pi} + N^{-\frac{4}{3}} \frac{s^2 \ln s}{8\pi} + \mathcal{O}\left((N^{-\frac{4}{3}} \ln N) s^2\right), \\ \left\langle \frac{1}{N} \text{tr} \phi^3 \right\rangle_0^{(1,0)} &= \frac{1024}{105\pi} + N^{-\frac{2}{3}} \frac{32s}{5\pi} + N^{-\frac{4}{3}} \frac{s^2}{\pi} + N^{-2} \frac{s^3 \ln s}{16\pi} + \mathcal{O}\left((N^{-2} \ln N) s^3\right), \\ \left\langle \frac{1}{N} \text{tr} \phi^5 \right\rangle_0^{(1,0)} &= \frac{8192}{315\pi} + N^{-\frac{2}{3}} \frac{512s}{21\pi} + N^{-\frac{4}{3}} \frac{8s^2}{\pi} + N^{-2} \frac{5s^3}{6\pi} + N^{-\frac{8}{3}} \frac{5s^4 \ln s}{128\pi} \\ &\quad + \mathcal{O}\left((N^{-\frac{8}{3}} \ln N) s^4\right), \\ &\dots \end{aligned} \quad (4.32)$$

at the planar level. The terms carrying the factor $\ln s$ are the leading nonanalytic terms that are regarded as universal contribution. As mentioned in (4.14), terms of

⁹This is similar to the planar case (4.26) that has been discussed in [1]. The variable ω used there corresponds to $N^{-\frac{2}{3}} \frac{s}{4}$.

order $(N^{-\frac{2}{3}(k+2)} \ln N) s^{k+2}$ are polynomials of s and nonuniversal. For higher-genus cases ($h = 1, 2, 3$), (4.6) with (4.25) leads to

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi \right\rangle_1^{(1,0)} &= -N^{\frac{2}{3}} \frac{1}{48\pi s} + \mathcal{O}(\ln(N^{-\frac{2}{3}} s)), & \left\langle \frac{1}{N} \text{tr} \phi^3 \right\rangle_1^{(1,0)} &= -\frac{\ln s}{32\pi} + \mathcal{O}((\ln N) s), \\ \left\langle \frac{1}{N} \text{tr} \phi^5 \right\rangle_1^{(1,0)} &= -\frac{1}{12\pi} - N^{-\frac{2}{3}} \frac{5s \ln s}{64\pi} + \mathcal{O}((N^{-\frac{2}{3}} \ln N) s), & \dots, \end{aligned} \quad (4.33)$$

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi \right\rangle_2^{(1,0)} &= -N^{\frac{8}{3}} \frac{1}{192\pi s^4} + \mathcal{O}(N^2 s^{-3}), & \left\langle \frac{1}{N} \text{tr} \phi^3 \right\rangle_2^{(1,0)} &= N^2 \frac{1}{384\pi s^3} + \mathcal{O}(N^{\frac{4}{3}} s^{-2}), \\ \left\langle \frac{1}{N} \text{tr} \phi^5 \right\rangle_2^{(1,0)} &= -N^{\frac{4}{3}} \frac{5}{1536\pi s^2} + \mathcal{O}(N^{\frac{2}{3}} s^{-1}), & \dots, \end{aligned} \quad (4.34)$$

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi \right\rangle_3^{(1,0)} &= -N^{\frac{14}{3}} \frac{5}{288\pi s^7} + \mathcal{O}(N^4 s^{-6}), & \left\langle \frac{1}{N} \text{tr} \phi^3 \right\rangle_3^{(1,0)} &= N^4 \frac{5}{1152\pi s^6} + \mathcal{O}(N^{\frac{10}{3}} s^{-5}), \\ \left\langle \frac{1}{N} \text{tr} \phi^5 \right\rangle_3^{(1,0)} &= -N^{\frac{10}{3}} \frac{5}{2304\pi s^5} + \mathcal{O}(N^{\frac{8}{3}} s^{-4}), & \dots. \end{aligned} \quad (4.35)$$

In (4.32)-(4.35), the leading nonanalytic contribution at $s = 0$ agrees with (4.17) and (4.18).

5 Instanton corrections to disk amplitudes

In this section, we compute instanton corrections to one-point functions at the planar level. In [3], it is shown that isolated eigenvalues around the origin give rise to nonperturbative effect and trigger spontaneous supersymmetry breaking. In fact, the origin is a saddle point of the effective potential for isolated eigenvalues in the large- N limit, and hence this configuration can be referred to as instantons¹⁰.

Here following the derivation in [17], let us compute instanton contribution to the one-point functions in the $(1,0)$ filling fraction. Namely, we are interested in the one-point functions in the presence of the instanton. The partition function in the $(1,0)$ sector is expressed by integrals along the positive real axis $\mathbf{R}_+ \equiv [0, \infty)$ with respect to all N eigenvalues. In the large- N limit, the perturbative partition function without instanton contribution comes from the integral region $[a, b]$ for each eigenvalue. In the decomposition of the partition function

$$Z_{(1,0)} = \sum_{p=0}^N Z_{(1,0)}|_{p\text{-inst.}}, \quad (5.1)$$

¹⁰It is well-known that such an isolated eigenvalue plays a role of nonperturbative effect in noncritical string theory [16, 17, 18, 19, 20, 21, 22].

the partition function with p instantons involved is defined as p eigenvalues integrated over the outside of $[a, b]$:

$$Z_{(1,0)}|_{p\text{-inst.}} = \binom{N}{p} \tilde{C}_N \int_a^b \prod_{i=1}^{N-p} d\lambda_i \int_{\mathbf{R}_+ \setminus [a,b]} \prod_{j=N-p+1}^N d\lambda_j \left(\prod_{n=1}^N 2\lambda_n \right) \Delta(\lambda^2)^2 \times e^{-N \sum_{i=1}^N \frac{1}{2}(\lambda_i - \mu^2)^2}. \quad (5.2)$$

The expectation value $\langle \mathcal{O} \rangle^{(1,0)}|_{p\text{-inst.}}$ of an operator \mathcal{O} under the partition function $Z_{(1,0)}|_{p\text{-inst.}}$ is defined accordingly. Then the expectation value of the operator \mathcal{O} under $Z_{(1,0)}$ can be written as

$$\langle \mathcal{O} \rangle^{(1,0)} = \sum_{p=0}^N \frac{Z_{(1,0)}|_{p\text{-inst.}}}{Z_{(1,0)}} \langle \mathcal{O} \rangle^{(1,0)}|_{p\text{-inst.}}. \quad (5.3)$$

From [3, 4], the partition functions behave as

$$Z_{(1,0)}|_{0\text{-inst.}} = 1, \quad Z_{(1,0)}|_{p\text{-inst.}} = \left(\frac{e^{-\frac{4}{3}s^{3/2}}}{16\pi s^{3/2}} \right)^p \times [1 + \mathcal{O}(s^{-3/2})] \quad (5.4)$$

in the double scaling limit with s finite but large. Hence the expansion in (5.3) by the instanton weight $e^{-\frac{4}{3}s^{3/2}}/(16\pi s^{3/2})$ becomes

$$\begin{aligned} \langle \mathcal{O} \rangle^{(1,0)} &= \langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \\ &+ Z_{(1,0)}|_{1\text{-inst.}} \left(\langle \mathcal{O} \rangle^{(1,0)}|_{1\text{-inst.}} - \langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \right) \\ &+ Z_{(1,0)}|_{2\text{-inst.}} \left(\langle \mathcal{O} \rangle^{(1,0)}|_{2\text{-inst.}} - \langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \right) \\ &+ (Z_{(1,0)}|_{1\text{-inst.}})^2 \left(-\langle \mathcal{O} \rangle^{(1,0)}|_{1\text{-inst.}} + \langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \right) \\ &+ Z_{(1,0)}|_{3\text{-inst.}} \left(\langle \mathcal{O} \rangle^{(1,0)}|_{3\text{-inst.}} - \langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \right) \\ &+ Z_{(1,0)}|_{1\text{-inst.}} Z_{(1,0)}|_{2\text{-inst.}} \left(-\langle \mathcal{O} \rangle^{(1,0)}|_{2\text{-inst.}} - \langle \mathcal{O} \rangle^{(1,0)}|_{1\text{-inst.}} + 2\langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \right) \\ &+ (Z_{(1,0)}|_{1\text{-inst.}})^3 \left(\langle \mathcal{O} \rangle^{(1,0)}|_{1\text{-inst.}} - \langle \mathcal{O} \rangle^{(1,0)}|_{0\text{-inst.}} \right) \\ &+ (\text{contribution from the total instanton number } p \geq 4). \end{aligned} \quad (5.5)$$

On the r.h.s., the third and fourth lines express contribution from the total instanton number two ($p = 2$), and the fifth, sixth and seventh lines from three ($p = 3$).

5.1 Schwinger-Dyson equations for ϕ^2 -resolvent at the presence of instantons

Let us consider the case where the number of instantons is $p = \mathcal{O}(N^0) \ll N$. Almost $(N - p)$ eigenvalues belong to the support $[a, b]$ that allows the usual $1/N$ or genus

expansion, whereas the remaining small number (p) of eigenvalues are outside of the support.

For a single-trace operator \mathcal{O} , we express the planar part (without handles, but with boundaries by instantons allowed) of $\langle \mathcal{O} \rangle_0^{(1,0)} \Big|_{p\text{-inst.}}$ as $\langle \mathcal{O} \rangle_0^{(1,0)} \Big|_{p\text{-inst.}}$. When $\langle \mathcal{O} \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} = \mathcal{O}(N^0)$ as usual, $\langle \mathcal{O} \rangle_0^{(1,0)} \Big|_{p\text{-inst.}}$ has contribution of $\mathcal{O}(N^0)$ from the $(N-p)$ eigenvalues and those of $\mathcal{O}(p/N)$ from the p eigenvalues. The latter is the deviation from the usual planar contribution due to the instantons. For the two-point planar contribution $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_0^{(1,0)}$, the large- N factorization holds even in the presence of the instantons:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \langle \mathcal{O}_1 \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} \langle \mathcal{O}_2 \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} \times (1 + \mathcal{O}(N^{-2})) \quad (5.6)$$

for $p = \mathcal{O}(N^0)$. Plugging (5.5) into the r.h.s. leads to the expansion of the two-point function by the instanton weight. The Schwinger-Dyson equation for the ϕ^2 -resolvent (1.2) derived in [1] reads

$$z \langle R_2(z) R_2(z) \rangle_0^{(1,0)} = (z^2 - \mu^2 z) \langle R_2(z) \rangle_0^{(1,0)} - z + \mu^2 - C_0 \quad (5.7)$$

with $C_0 = \langle \frac{1}{N} \text{tr} \phi^2 \rangle_0^{(1,0)}$. From the expansion (5.5) for $\langle R_2(z) \rangle_0^{(1,0)}$ and C_0 , and from (5.6) with $\mathcal{O}_1 = \mathcal{O}_2 = R_2(z)$, we have the expansion of (5.7) by the instanton weight:

0-instanton sector:

$$z \left(\langle R_2(z) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} \right)^2 = (z^2 - \mu^2 z) \langle R_2(z) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} - z + \mu^2 - C_0|_{0\text{-inst.}}, \quad (5.8)$$

1-instanton sector:

$$\begin{aligned} & z \left\{ \left(\langle R_2(z) \rangle_0^{(1,0)} \Big|_{1\text{-inst.}} \right)^2 - \left(\langle R_2(z) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} \right)^2 \right\} \\ &= (z^2 - \mu^2 z) \left\{ \langle R_2(z) \rangle_0^{(1,0)} \Big|_{1\text{-inst.}} - \langle R_2(z) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} \right\} - (C_0|_{1\text{-inst.}} - C_0|_{0\text{-inst.}}). \end{aligned} \quad (5.9)$$

Plugging (5.8) into (5.9) simplifies the equation as

$$z \left(\langle R_2(z) \rangle_0^{(1,0)} \Big|_{1\text{-inst.}} \right)^2 = (z^2 - \mu^2 z) \langle R_2(z) \rangle_0^{(1,0)} \Big|_{1\text{-inst.}} - z + \mu^2 - C_0|_{1\text{-inst.}}. \quad (5.10)$$

For higher instantons, we can reduce the equations in a similar manner to obtain

p -instanton sector:

$$z \left(\langle R_2(z) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} \right)^2 = (z^2 - \mu^2 z) \langle R_2(z) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} - z + \mu^2 - C_0|_{p\text{-inst.}}, \quad (5.11)$$

which is solved by

$$\langle R_2(z) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \frac{1}{2} \left[z - \mu^2 - \sqrt{(z - \mu^2)^2 - 4 + \frac{4(\mu^2 - C_0|_{p\text{-inst.}})}{z}} \right]. \quad (5.12)$$

In $p = 0$ case, C_0 is determined by requiring no singularity other than the cut $[a^2, b^2]$ that corresponds to the perturbative saddle points. Thus

$$C_0|_{0\text{-inst.}} = \mu^2. \quad (5.13)$$

From (5.12), we see that the eigenvalues relevant to instantons are around the origin, i.e.

$$\oint_{0,z} \langle R_2(z) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \frac{p}{N}. \quad (5.14)$$

Note that this equation indicates that $\mu^2 - C_0|_{p\text{-inst.}}$ is of $\mathcal{O}(1/N)$ and hence from the factorization (5.6), the expression (5.12) is valid within the linear order of $\mu^2 - C_0|_{p\text{-inst.}}$:

$$\langle R_2(z) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \langle R_2(z) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} - \frac{\mu^2 - C_0|_{p\text{-inst.}}}{z\sqrt{(z - \mu^2)^2 - 4}} + \mathcal{O}((\mu^2 - C_0|_{p\text{-inst.}})^2). \quad (5.15)$$

In imposing the condition (5.14), we take care of the branch cut of $\sqrt{(z - \mu^2)^2 - 4} = \sqrt{(z - a^2)(z - b^2)}$ and see $\sqrt{(z - \mu^2)^2 - 4} \Big|_{z=0} = -ab = -\sqrt{\mu^4 - 4}$. Then the solution becomes

$$C_0|_{p\text{-inst.}} = \mu^2 - \frac{p}{N} \sqrt{\mu^4 - 4} + \mathcal{O}((p/N)^2), \quad (5.16)$$

$$\langle R_2(z) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \langle R_2(z) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} - \frac{p}{N} \frac{\sqrt{\mu^4 - 4}}{z\sqrt{(z - \mu^2)^2 - 4}} + \mathcal{O}((p/N)^2). \quad (5.17)$$

$C|_{p\text{-inst.}} - \mu^2 = \langle \frac{1}{N} \text{tr}(\phi^2 - \mu^2) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}}$ can also be obtained from (5.4) as

$$C|_{p\text{-inst.}} - \mu^2 = N^{-2} \frac{\partial}{\partial \mu^2} \ln Z_{(1,0)} \Big|_{p\text{-inst.}} = -N^{-\frac{4}{3}} 2ps^{\frac{1}{2}} (1 + \mathcal{O}(s^{-4/3})) \quad (5.18)$$

in the double scaling limit with s finite but large, which is consistent with (5.16) up to higher-genus contribution in the last factor $(1 + \mathcal{O}(s^{-4/3}))$.

5.2 Instanton corrections to one-point functions

Now, instanton corrections to the disk amplitudes $\langle \frac{1}{N} \text{tr} \phi^n \rangle_0^{(1,0)}$ ($n \in \mathbf{N}$) are computed as in (3.3):¹¹

$$\left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \oint_{0 \cup [a, b], z} 2z \cdot z^n \langle R_2(z^2) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}}. \quad (5.19)$$

The integral encircling the origin vanishes due to the factor $2z \cdot z^n$, and we have

$$\left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} \Big|_{0\text{-inst.}} + \frac{p}{N} \sqrt{\mu^4 - 4} I_{-1, n-2} + \mathcal{O}((p/N)^2) \quad (5.20)$$

¹¹We recall the footnote 5.

by setting $x = \mu^2 - z^2$ and (4.7). From (4.25),

$$I_{-1, n-2} = b^{n-2} F\left(1 - \frac{n}{2}, \frac{1}{2}, 1; \frac{4}{b^2}\right). \quad (5.21)$$

For odd n , the first few expressions of $I_{-1, n-2}$ are given by

$$\begin{aligned} I_{-1, -1} &= -\frac{1}{2\pi} \ln s + \mathcal{O}((\ln N)s^0), \\ I_{-1, 1} &= \frac{4}{\pi} - N^{-\frac{2}{3}} \frac{1}{4\pi} s \ln s + \mathcal{O}((N^{-\frac{2}{3}} \ln N)s), \\ I_{-1, 3} &= \frac{32}{3\pi} + N^{-\frac{2}{3}} \frac{6}{\pi} s - N^{-\frac{4}{3}} \frac{3}{16\pi} s^2 \ln s + \mathcal{O}((N^{-\frac{4}{3}} \ln N)s^2), \\ &\dots \end{aligned} \quad (5.22)$$

The leading nonanalytic term of $I_{-1, 2k-1}$ reads

$$I_{-1, 2k-1} = -\left(N^{-\frac{2}{3}}\right)^k \frac{1}{2\pi^{\frac{3}{2}}} \frac{\Gamma(k + \frac{1}{2})}{k!} s^k \ln s + (\text{less singular at } s = 0), \quad (5.23)$$

which is consistent with (4.12). For even n , $I_{-1, n-2}$ reduces to a polynomial of $b^2 (= 4 + N^{-\frac{2}{3}}s)$:

$$\begin{aligned} I_{-1, 0} &= 1, \\ I_{-1, 2} &= 2 + N^{-\frac{2}{3}}s, \\ I_{-1, 4} &= 6 + N^{-\frac{2}{3}}4s + N^{-\frac{4}{3}}s^2, \\ I_{-1, 6} &= 20 + N^{-\frac{2}{3}}18s + N^{-\frac{4}{3}}6s^2 + N^{-2}s^3, \\ &\dots \end{aligned} \quad (5.24)$$

Thus the difference of the p -instanton contribution from the 0-instanton one:

$$\Delta \left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} \equiv \left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} - \left\langle \frac{1}{N} \text{tr} \phi^n \right\rangle_0^{(1,0)} \Big|_{0\text{-inst.}} \quad (5.25)$$

becomes

$$\begin{aligned} \Delta \left\langle \frac{1}{N} \text{tr} \phi \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= N^{-\frac{4}{3}}p \left[-\frac{1}{\pi} s^{\frac{1}{2}} \ln s + \mathcal{O}((\ln N)s^{\frac{1}{2}}) \right] + \mathcal{O}((p/N)^2), \\ \Delta \left\langle \frac{1}{N} \text{tr} \phi^3 \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= N^{-\frac{4}{3}}p \left[\frac{8}{\pi} s^{\frac{1}{2}} - N^{-\frac{2}{3}} \frac{1}{2\pi} s^{\frac{3}{2}} \ln s + \mathcal{O}((N^{-\frac{2}{3}} \ln N)s^{\frac{3}{2}}) \right] + \mathcal{O}((p/N)^2), \\ \Delta \left\langle \frac{1}{N} \text{tr} \phi^5 \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= N^{-\frac{4}{3}}p \left[\frac{64}{3\pi} s^{\frac{1}{2}} + N^{-\frac{2}{3}} \frac{44}{3\pi} s^{\frac{3}{2}} - N^{-\frac{4}{3}} \frac{3}{8\pi} s^{\frac{5}{2}} \ln s + \mathcal{O}((N^{-\frac{4}{3}} \ln N)s^{\frac{5}{2}}) \right] \\ &\quad + \mathcal{O}((p/N)^2), \\ &\dots, \end{aligned} \quad (5.26)$$

and

$$\begin{aligned}
\Delta \left\langle \frac{1}{N} \text{tr} \phi^2 \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= N^{-\frac{4}{3}} p \left[2s^{\frac{1}{2}} + N^{-\frac{2}{3}} \frac{1}{4} s^{\frac{3}{2}} + \mathcal{O} \left(N^{-\frac{4}{3}} s^{\frac{5}{2}} \right) \right] + \mathcal{O}((p/N)^2), \\
\Delta \left\langle \frac{1}{N} \text{tr} \phi^4 \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= N^{-\frac{4}{3}} p \left[4s^{\frac{1}{2}} + N^{-\frac{2}{3}} \frac{5}{2} s^{\frac{3}{2}} + \mathcal{O} \left(N^{-\frac{4}{3}} s^{\frac{5}{2}} \right) \right] + \mathcal{O}((p/N)^2), \\
\Delta \left\langle \frac{1}{N} \text{tr} \phi^6 \right\rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= N^{-\frac{4}{3}} p \left[12s^{\frac{1}{2}} + N^{-\frac{2}{3}} \frac{19}{2} s^{\frac{3}{2}} + N^{-\frac{4}{3}} \frac{93}{32} s^{\frac{5}{2}} + \mathcal{O} \left(N^{-2} s^{\frac{7}{2}} \right) \right] \\
&\quad + \mathcal{O}((p/N)^2), \\
&\dots
\end{aligned} \tag{5.27}$$

5.3 Operator mixing

For odd n , we expect from (5.23) that the terms with the $\ln s$ factor would have a universal meaning in (5.26), which reads

$$- \left(N^{-\frac{2}{3}} \right)^{k+2} \frac{p}{\pi^{\frac{3}{2}}} \frac{\Gamma(k + \frac{1}{2})}{k!} s^{k+\frac{1}{2}} \ln s \tag{5.28}$$

with $n = 2k + 1$. In fact, the power of N is the same as that in the perturbative result (4.19), which shows that nonperturbative as well as perturbative contribution equally survive in the double scaling limit as should be. On the other hand, polynomials of s in (5.22) appear as nonanalytic terms of half-integer powers due to the factor $\sqrt{\mu^4 - 4} = N^{-\frac{1}{3}} 2s^{\frac{1}{2}} \left[1 + \mathcal{O}(N^{-\frac{2}{3}} s) \right]$ in the one-point functions (5.26). Since their nonanalyticity is stronger than the would-be universal term of $s^{k+\frac{1}{2}} \ln s$ at $s = 0$, there seems no reason to throw away such terms. This is in contrast to the perturbative case where we can safely discard polynomials of s in (4.32) as nonuniversal parts.

In [1], we have encountered similar difficulty in computing cylinder amplitudes and discussed the operator mixing to resolve it. The operator mixing in [1] is given as

$$\begin{aligned}
\Phi_1 &= \frac{1}{N} \text{tr} \phi, \\
\Phi_3 &= \frac{1}{N} \text{tr} \phi^3 - \frac{4}{\pi} \left(1 + \bar{\alpha}_{3,2}^{(1)} \omega + \mathcal{O}(\omega^2) \right) \frac{1}{N} \text{tr} \phi^2, \\
\Phi_5 &= \frac{1}{N} \text{tr} \phi^5 - \frac{4}{\pi} \left(1 + \bar{\alpha}_{5,4}^{(1)} \omega + \mathcal{O}(\omega^2) \right) \frac{1}{N} \text{tr} \phi^4 \\
&\quad - \frac{8}{3\pi} \left(1 + 3(1 - \bar{\alpha}_{5,4}^{(1)}) \omega + \mathcal{O}(\omega^2) \right) \frac{1}{N} \text{tr} \phi^2
\end{aligned} \tag{5.29}$$

with $\omega = N^{-\frac{2}{3}} \frac{s}{4}$ for the $(1, 0)$ filling fraction. $\bar{\alpha}_{3,2}^{(1)}$ and $\bar{\alpha}_{5,4}^{(1)}$ are numerical constants undetermined from the cylinder amplitudes among Φ_1 , Φ_3 and Φ_5 . Since the one-point

functions of even-power operators (5.27) have half-integer powers of s , it is reasonable to expect that the operator mixing cancels the half-integer powers between (5.26) and (5.27). Straightforward calculations actually prove that is the case, leading to

$$\begin{aligned}\Delta \langle \Phi_1 \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= \left(N^{-\frac{2}{3}}\right)^2 p \left[-\frac{1}{\pi} s^{\frac{1}{2}} \ln s + \mathcal{O}\left((\ln N) s^{\frac{1}{2}}\right) \right] + \mathcal{O}\left((p/N)^2\right), \\ \Delta \langle \Phi_3 \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= \left(N^{-\frac{2}{3}}\right)^3 p \left[-\frac{1}{2\pi} s^{\frac{3}{2}} \ln s + \mathcal{O}\left((\ln N) s^{\frac{3}{2}}\right) \right] + \mathcal{O}\left((p/N)^2\right), \\ \Delta \langle \Phi_5 \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= \left(N^{-\frac{2}{3}}\right)^4 p \left[-\frac{3}{8\pi} s^{\frac{5}{2}} \ln s + \mathcal{O}\left((\ln N) s^{\frac{5}{2}}\right) \right] + \mathcal{O}\left((p/N)^2\right). \quad (5.30)\end{aligned}$$

In particular, $\bar{\alpha}_{3,2}^{(1)}$ and $\bar{\alpha}_{5,4}^{(1)}$ cancel in the leading terms in (5.30) and remain undetermined again. Similarly to the perturbative case, the last terms of $\mathcal{O}\left((\ln N) s^{k+\frac{1}{2}}\right)$ in the square brackets are less singular at $s = 0$ than the first terms. Hence the operator mixing discussed in the cylinder amplitudes (5.29) works even at the nonperturbative instanton contribution.

Thus we confirm again that the double scaling limit keeps valid even in this case. By multiplying the “wave function renormalization” factor $N^{\frac{2}{3}(k+2)}$, (5.30) becomes finite in this limit¹². The one-point functions in the presence of instantons also has the logarithmic singularity. The s -dependence of $\Delta \langle \Phi_{2k+1} \rangle_0^{(1,0)} \Big|_{p\text{-inst.}}$ is $s^{k+\frac{1}{2}} \ln s$, which is different from the planar result of the zero-instanton sector $s^{k+2} \ln s$. The difference of the power $s^{-\frac{3}{2}}$ is proportional to g_s , and it can be interpreted as contribution from a hole created by the instanton to the one-point function¹³. This result would be important in trying to identify a counterpart of the matrix model instanton in the type IIA side. In appendix C, we present other derivation of (5.17) based on distortion of the eigenvalue distribution by instantons.

6 Correlation functions in the full sector

So far we have considered the correlation functions with a definite filling fraction, say $(\nu_+, \nu_-) = (1, 0)$. In this section we discuss those in the full sector, namely summed over the filling fractions. At first sight, it seems difficult to formulate them because of the vanishing total partition function (2.10). In order to regularize it and get well-defined correlation functions, we introduce a factor $e^{-i\alpha\nu_-N}$ with a small parameter α in front of $Z_{(\nu_+, \nu_-)}$ [3]:

$$Z_\alpha \equiv \sum_{\nu_-N=0}^N \frac{N!}{(\nu_+N)!(\nu_-N)!} e^{-i\alpha\nu_-N} Z_{(\nu_+, \nu_-)} = (1 - e^{-i\alpha})^N Z_{(1,0)}. \quad (6.1)$$

¹²This is also the case with the $c = 0$ bosonic string theory as discussed in [17].

¹³Note that backreaction to the instantons, i.e. influence of the presence of the operator on the instanton background, is not taken into account in this calculation.

Correspondingly, regularized correlation functions among K single-trace operators are

$$\begin{aligned}
\left\langle \prod_{a=1}^K \frac{1}{N} \text{tr } \mathcal{O}_a(\phi) \right\rangle_\alpha &\equiv \frac{\tilde{C}_N}{Z_\alpha} \sum_{\nu_-=N=0}^N \frac{N!}{(\nu_+ N)! (\nu_- N)!} e^{-i\alpha \nu_- N} \\
&\times \int_0^\infty \left(\prod_{i=1}^{\nu_+ N} 2\lambda_i d\lambda_i \right) \int_{-\infty}^0 \left(\prod_{j=\nu_+ N+1}^N 2\lambda_j d\lambda_j \right) \Delta(\lambda^2) \\
&\times \left(\prod_{a=1}^K \frac{1}{N} \sum_{i=1}^N \mathcal{O}_a(\lambda_i) \right) e^{-N \sum_{m=1}^N \frac{1}{2} (\lambda_m^2 - \mu^2)^2}. \quad (6.2)
\end{aligned}$$

As discussed in [10, 11], the regularization parameter α could also be interpreted as an external field in discussing spontaneous supersymmetry breaking, e.g. the magnetic field in the spontaneous magnetization in spin systems. In [3], the one-point function $\langle \frac{1}{N} \text{tr } (iB) \rangle_\alpha$, equivalently $\langle \frac{1}{N} \text{tr } (\phi^2 - \mu^2) \rangle_\alpha$, has been computed as one of the order parameters of the supersymmetry breaking. There, the result is independent of α and has a well-defined limit for $\alpha \rightarrow 0$. In general, when all of the operators $\frac{1}{N} \text{tr } \mathcal{O}_a$ ($a = 1, \dots, K$) are even for the sign flip $\lambda_j \rightarrow -\lambda_j$ ($j = \nu_+ N + 1, \dots, N$) considered in (2.9), the α -dependence between the numerator and the denominator cancels each other in (6.2). Namely, the regularization works for correlators among even-power operators. On the other hand, this is not the case with odd-power operators $\frac{1}{N} \text{tr } \phi^{2k+1}$ ($k \in \mathbf{N} \cup \{0\}$) involved. In what follows, we argue that nontrivial α -dependence appearing in correlation functions of the odd-power operators can be absorbed into a “wave-function renormalization” and then the limit $\alpha \rightarrow 0$ can be safely taken to reduce to the correlation functions to those in the $(1, 0)$ filling fraction.

6.1 One-point functions

The odd-power operator $\frac{1}{N} \text{tr } \phi^{2k+1}$ changes under the sign flip as

$$\frac{1}{N} \sum_{i=1}^N \lambda_i^{2k+1} \rightarrow \frac{1}{N} \sum_{i=1}^{\nu_+ N} \lambda_i^{2k+1} - \frac{1}{N} \sum_{j=\nu_+ N+1}^N \lambda_j^{2k+1} \quad (6.3)$$

in terms of the eigenvalues. By using permutation symmetries with respect to $\lambda_1, \dots, \lambda_N$ in the eigenvalue-integrals ((6.2) with $K = 1$ and $\mathcal{O}_1(\phi) = \phi^{2k+1}$), the r.h.s. of (6.3) can be replaced by $(\nu_+ - \nu_-)\lambda_1$ and further by $(\nu_+ - \nu_-)\frac{1}{N} \sum_{i=1}^N \lambda_i$ in the integrals. Then, the one-point function $\langle \frac{1}{N} \text{tr } \phi^{2k+1} \rangle_\alpha$ becomes

$$\left\langle \frac{1}{N} \text{tr } \phi^{2k+1} \right\rangle_\alpha = \frac{1}{(1 - e^{-i\alpha})^N} \left\{ \sum_{\nu_-=N=0}^N \frac{N! (\nu_+ - \nu_-)}{(\nu_+ N)! (\nu_- N)!} (-e^{-i\alpha})^{\nu_- N} \right\} \left\langle \frac{1}{N} \text{tr } \phi^{2k+1} \right\rangle^{(1,0)}. \quad (6.4)$$

The sum in the curly bracket on the r.h.s. is computed as

$$\begin{aligned} \sum_{n=0}^N \frac{N!}{n!(N-n)!} \left(1 - 2\frac{n}{N}\right) (-e^{-i\alpha})^n &= \left(1 - \frac{2i}{N}\partial_\alpha\right) (1 - e^{-i\alpha})^N \\ &= (1 - e^{-i\alpha})^N \left\{1 + \frac{2e^{-i\alpha}}{1 - e^{-i\alpha}}\right\}. \end{aligned} \quad (6.5)$$

Thus we find

$$\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle_\alpha = C(\alpha) \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \right\rangle^{(1,0)} \quad (6.6)$$

with

$$C(\alpha) \equiv -i \cot \frac{\alpha}{2}. \quad (6.7)$$

Although $C(\alpha)$ diverges as $\alpha \rightarrow 0$, (6.6) seems to suggest that the divergence could be absorbed into a kind of “wave function renormalization”

$$\frac{1}{N} \text{tr} \phi^{2k+1} \rightarrow C(\alpha)^{-1} \frac{1}{N} \text{tr} \phi^{2k+1} \quad (6.8)$$

in computing correlation functions in the full sector. Then the result is reduced to the one in the $(1, 0)$ filling fraction that is finite and well-defined as $\alpha \rightarrow 0$. Of course, we need to check whether (6.8) is valid or not in other cases. Let us consider the two-point functions of odd-power operators as the first nontrivial check.

6.2 Two-point functions

As in the case of the one-point functions, we first consider the sign flip in the product of the two odd-power operators $\frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1}$. In terms of the eigenvalues, it changes to

$$\left(\frac{1}{N} \sum_{i_1=1}^{\nu_+ N} \lambda_{i_1}^{2k+1} - \frac{1}{N} \sum_{j_1=\nu_+ N+1}^N \lambda_{j_1}^{2k+1} \right) \left(\frac{1}{N} \sum_{i_2=1}^{\nu_+ N} \lambda_{i_2}^{2\ell+1} - \frac{1}{N} \sum_{j_2=\nu_+ N+1}^N \lambda_{j_2}^{2\ell+1} \right). \quad (6.9)$$

Expanding the product and extracting terms with $i_1 = i_2$ or $j_1 = j_2$ leads to

$$\begin{aligned} & \frac{1}{N^2} \sum_{i=1}^{\nu_+ N} \lambda_i^{2k+1} \lambda_i^{2\ell+1} + \frac{1}{N^2} \sum_{j=\nu_+ N+1}^N \lambda_j^{2k+1} \lambda_j^{2\ell+1} + \frac{1}{N^2} \sum_{i_1 \neq i_2} \lambda_{i_1}^{2k+1} \lambda_{i_2}^{2\ell+1} + \frac{1}{N^2} \sum_{j_1 \neq j_2} \lambda_{j_1}^{2k+1} \lambda_{j_2}^{2\ell+1} \\ & - \frac{1}{N^2} \sum_{i=1}^{\nu_+ N} \sum_{j=\nu_+ N+1}^N (\lambda_i^{2k+1} \lambda_j^{2\ell+1} + \lambda_j^{2k+1} \lambda_i^{2\ell+1}), \end{aligned} \quad (6.10)$$

where the sum of i_1 and i_2 (j_1 and j_2) is understood to run from 1 to $\nu_+ N$ (from $\nu_+ N + 1$ to N). Use of the permutation symmetry of the eigenvalue-integrals allows us to replace

this by ¹⁴

$$(\nu_+ - \nu_-)^2 \lambda_1^{2k+1} \lambda_2^{2\ell+1} + \frac{1}{2N} (\lambda_1^{2k+1} - \lambda_2^{2k+1}) (\lambda_1^{2k+1} - \lambda_2^{2\ell+1}) \quad (6.11)$$

in the integrals. Then we have

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle_\alpha &= \frac{1}{(1 - e^{-i\alpha})^N} \sum_{\nu_- = 0}^N \frac{N!}{(\nu_+ N)! (\nu_- N)!} (-e^{-i\alpha})^{\nu_- N} \\ &\times \left[(\nu_+ - \nu_-)^2 \langle \lambda_1^{2k+1} \lambda_2^{2\ell+1} \rangle^{(1,0)} + \frac{1}{2N} \langle (\lambda_1^{2k+1} - \lambda_2^{2k+1}) (\lambda_1^{2k+1} - \lambda_2^{2\ell+1}) \rangle_C^{(1,0)} \right]. \end{aligned} \quad (6.12)$$

Note that

$$\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle^{(1,0)} = \langle \lambda_1^{2k+1} \lambda_2^{2\ell+1} \rangle^{(1,0)} + \frac{1}{2N} \langle (\lambda_1^{2k+1} - \lambda_2^{2k+1}) (\lambda_1^{2k+1} - \lambda_2^{2\ell+1}) \rangle_C^{(1,0)} \quad (6.13)$$

and the second term on the r.h.s. is negligible compared to the first term in the double scaling limit. Thus under the prescription in taking the limits:

1. take the double scaling limit first,
2. then, turn off α ,

we obtain

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle_\alpha &= \frac{1}{(1 - e^{-i\alpha})^N} \left\{ \sum_{n=0}^N \frac{N!}{n! (N-n)!} \left(1 - 2\frac{n}{N}\right)^2 (-e^{-i\alpha})^n \right\} \\ &\times \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle^{(1,0)}. \end{aligned} \quad (6.14)$$

After computing the sum, extracting the connected pieces from this expression leads to

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle_{\alpha, C} &= C(\alpha)^2 \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle_C^{(1,0)} \\ &- \frac{1}{N} (C(\alpha)^2 - 1) \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle^{(1,0)}. \end{aligned} \quad (6.15)$$

The two-point function in the last term consists of the connected and disconnected pieces: $\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \rangle_C^{(1,0)}$ and $\langle \frac{1}{N} \text{tr} \phi^{2k+1} \rangle^{(1,0)} \langle \frac{1}{N} \text{tr} \phi^{2\ell+1} \rangle^{(1,0)}$. They are of the same order in the double scaling limit as far as the universal parts are concerned. Namely, we assume that in the double scaling limit, non-universal parts which would become dominant in each correlation function are subtracted in advance. Then the last term on the r.h.s. of

¹⁴This manipulation is similar to the one presented in appendix B of [1].

(6.15) can be neglected in the double scaling limit due to the prefactor $\frac{1}{N}$. In conclusion, we arrive at

$$\left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle_{\alpha, C} \bigg|_{\text{univ.}} = C(\alpha)^2 \left\langle \frac{1}{N} \text{tr} \phi^{2k+1} \frac{1}{N} \text{tr} \phi^{2\ell+1} \right\rangle_C^{(1,0)} \bigg|_{\text{univ.}}, \quad (6.16)$$

concerning the universal parts in the double scaling limit. Thus we again find that the two-point functions of the “renormalized” operators $C(\alpha)^{-1} \frac{1}{N} \text{tr} \phi^{2k+1}$ are independent of α and reduced to those in the $(\nu_+, \nu_-) = (1, 0)$ sector in the $\alpha \rightarrow 0$ limit.

In summary, the one- and two-point correlation functions of “renormalized” odd-power operators

$$\widehat{\Phi}_k = \frac{1}{C(\alpha)} N^{\frac{2}{3}(k+2)} \left(\frac{1}{N} \text{tr} \phi^{2k+1} - (\text{nonuniversal parts}) \right) \quad (6.17)$$

are all finite in the prescription of the limit where we take first the double scaling limit, then $\alpha \rightarrow 0$ limit. In the above equation, “(nonuniversal parts)” indicates both of the mixing terms mentioned in (2.14) and nonuniversal parts of $\frac{1}{N} \text{tr} \phi^{2k+1}$ itself. They may be more dominant than the universal part in the double scaling limit unless we subtract it in advance. Thus the argument in this section validates concentrating on correlation functions in the $(1, 0)$ sector.

7 Discussions

In this paper, we have computed one-point functions of the operators $\frac{1}{N} \text{tr} \phi^n$ ($n \in \mathbf{N}$) to all order of genus expansion and their instanton contribution in the supersymmetric double-well matrix model, which extends the work of correlation functions at the planar level [1]. The matrix model is proposed to describe two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond background [2]. The operators with even n (even-power operators) are protected by supersymmetry, while those with odd n (odd-power operators) are not. We have seen that this difference is reflected by qualitatively different behavior in the correlation functions. For example, genus-expansion of the even n case terminates at some order, whereas the odd n case yields non-Borel summable series. The divergence is due to the coefficients of the series growing as $(2h)!$ for a large genus h , which has been recognized as a characteristic feature of string perturbation series [15]. This indicates that operators unprotected by supersymmetry play an essential role to understand superstring theory from the corresponding matrix model. From the non-Borel summable asymptotic series, we can read nonperturbative ambiguity that turns out to be of the same order as instanton effects found in [3, 4]. The idea of resurgence suggests that the ambiguity from the perturbative series is canceled with one arising from fluctuations around the instanton background (for example, see [22, 23, 24, 25]). It is intriguing to check whether it works as well in our matrix model or superstring theory with its target supersymmetry spontaneously broken [26].

It is discussed in [1, 2] that single-trace operators with operator mixing in our matrix model corresponds to integrated vertex operators in the type IIA superstring theory. The explicit form of the operator mixing is presented there based on the result of planar two-point (cylinder) amplitudes. We have seen here that the operator mixing is also consistent with nonperturbative instanton contribution to the one-point functions. In addition, the difference of the filling fraction ($\nu_+ - \nu_-$) in the matrix model is proportional to the strength of the Ramond-Ramond background flux [1, 2]. Although the correlation functions have been computed at a fixed sector of the filling fraction, typically the $(1, 0)$ sector here, we have shown that the computation by the total partition function summed over the filling fractions is regularized by a “wave-function renormalization” factor and yields the same result as in the $(1, 0)$ sector. It would be interesting to consider the meaning of the regularization in the type IIA superstring side, which may give new insight to the structure of vacua in the superstring theory.

In the next papers [5, 26], we will present the computation of two-point functions in the matrix model to all order in genus expansion, and discuss the further consistency of the operator mixing and resurgence.

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A Solution of the recursion relation (4.4)

In this appendix, we present a solution of the recursion relation (4.4).

A.1 $C_{j, 3j-1}$, $C_{j, 2j}$

In the case $r = 3j + 2$, by noting (4.5), the recursion relation (4.4) is reduced to

$$\frac{1}{(6j+3)!!} C_{j+1, 3j+2} = \frac{2}{3} \frac{1}{j+1} \frac{1}{(6j-3)!!} C_{j, 3j-1}. \quad (\text{A.1})$$

In terms of $D_j \equiv \frac{1}{(6j-3)!!} C_{j, 3j-1}$, it is easy to solve this as

$$D_{j+1} = \frac{2}{3} \frac{1}{j+1} D_j = \cdots = \left(\frac{2}{3}\right)^j \frac{1}{(j+1)!} D_1. \quad (\text{A.2})$$

From $D_1 = \frac{1}{3!!}C_{1,2} = \frac{1}{3}$, we have

$$D_j = \frac{1}{2} \left(\frac{2}{3} \right)^j \frac{1}{j!}, \quad (\text{A.3})$$

and thus

$$C_{j,3j-1} = \frac{1}{2} \left(\frac{2}{3} \right)^j \frac{(6j-3)!!}{j!} \quad (j \in \mathbf{N}). \quad (\text{A.4})$$

In the case of $r = 2j + 2$, we can similarly obtain

$$\frac{C_{j+1,2j+2}}{(4j+3)!!} = \frac{2j+1}{2j+3} \frac{C_{j,2j}}{(4j-1)!!} \quad (\text{A.5})$$

which leads to a solution:

$$C_{j,2j} = \frac{(4j-1)!!}{2j+1} \quad (j \in \mathbf{N}). \quad (\text{A.6})$$

A.2 $C_{j,2j+1}$, $C_{j,2j+2}$, $C_{j,2j+3}$

For $r = 2j + 3$, the recursion relation (4.4) becomes

$$\frac{j+2}{(4j+5)!!} C_{j+1,2j+3} = \frac{j+1}{(4j+1)!!} C_{j,2j+1} + \frac{1}{2j+1}, \quad (\text{A.7})$$

where we have used (A.6). By considering $E_j \equiv \frac{j+1}{(4j+1)!!} C_{j,2j+1}$, we have

$$E_j = \sum_{\ell=2}^j \frac{1}{2\ell-1}. \quad (\text{A.8})$$

Therefore,

$$\begin{aligned} C_{j,2j+1} &= \frac{(4j+1)!!}{j+1} \sum_{\ell=2}^j \frac{1}{2\ell-1} \quad (j \geq 2), \\ C_{1,3} &= 0. \end{aligned} \quad (\text{A.9})$$

Repeating a similar procedure for $r = 2j + 4$, $2j + 5$, we obtain

$$\begin{aligned} C_{j,2j+2} &= 2 \frac{(4j+3)!!}{2j+3} \sum_{\ell'=2}^{j-1} \frac{1}{\ell'+1} \sum_{\ell=2}^{\ell'} \frac{1}{2\ell-1} \quad (j \geq 3), \\ C_{1,4} &= C_{2,6} = 0, \end{aligned} \quad (\text{A.10})$$

and

$$\begin{aligned} C_{j,2j+3} &= 2 \frac{(4j+5)!!}{j+2} \sum_{\ell''=2}^{j-2} \frac{1}{2\ell''+5} \sum_{\ell'=2}^{\ell''} \frac{1}{\ell'+1} \sum_{\ell=2}^{\ell'} \frac{1}{2\ell-1} \quad (j \geq 4), \\ C_{1,5} &= C_{2,7} = C_{3,9} = 0. \end{aligned} \quad (\text{A.11})$$

A.3 $C_{j,r}$ for general r

From the expressions of (A.6), (A.9), (A.10), and (A.11), we can find out the form of $C_{j,r}$ for general r :

$$\begin{aligned}
C_{j,2j+r} &= \frac{1}{2} \frac{(4j+2r-1)!!}{j+\frac{r+1}{2}} \sum_{\ell_r=2}^{j-r+1} \frac{1}{\ell_r+\frac{3r-4}{2}} \sum_{\ell_{r-1}=2}^{\ell_r} \frac{1}{\ell_{r-1}+\frac{3r-7}{2}} \times \cdots \\
&\quad \times \sum_{\ell_2=2}^{\ell_3} \frac{1}{\ell_2+1} \sum_{\ell_1=2}^{\ell_2} \frac{1}{\ell_1-\frac{1}{2}} \quad (1 \leq r \leq j-1), \\
C_{j,2j} &= \frac{1}{2} \frac{(4j-1)!!}{j+\frac{1}{2}}, \tag{A.12}
\end{aligned}$$

and all the others vanish.

In fact, when $r = j - 1$ in (A.12), each of the ℓ_i ($i = 1, 2, \dots, j - 1$) appearing in the sum takes the value 2 alone, and the expression reproduces (A.4).

Finally, we explicitly present the first several nonvanishing expressions for $C_{j,r}$:

$$\begin{aligned}
C_{1,2} &= 1, & C_{2,4} &= 21, & C_{2,5} &= 105, \\
C_{3,6} &= 1485, & C_{3,7} &= 18018, & C_{3,8} &= 50050, \tag{A.13}
\end{aligned}$$

which agree with the result given in [12].

B Other derivation of one-point functions of even-power operators

In this appendix, we compute one-point functions of the even-power operators $\frac{1}{N} \text{tr} \phi^{2\ell}$ or $\frac{1}{N} \text{tr} B^\ell$ ($\ell \in \mathbf{N}$) at arbitrary genus in a different manner from the text. Since these are independent of the sector of the filling fraction as discussed in [1], let us focus on the $(1, 0)$ filling fraction case. By diagonalizing ϕ as $\phi = U \Lambda U^\dagger$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, the partition function can be written as

$$Z_{(1,0)} = \tilde{C}_N \int d^{N^2} B \int_0^\infty \prod_{i=1}^N (d\lambda_i W''(\lambda_i)) \prod_{i>j} (W'(\lambda_i) - W'(\lambda_j))^2 e^{-N \text{tr} (\frac{1}{2} B^2 + i B W'(\Lambda))} \tag{B.1}$$

with $W'(x) = x^2 - \mu^2$. Note that the following argument is valid in a more general superpotential as long as $Z_{(1,0)}$ does not vanish. For example, when $W'(x)$ is a polynomial of the odd degree, the total partition function remains nonzero. The argument below (B.4) is nonperturbatively correct for that case, with the replacement of $Z_{(1,0)}|_{\text{pert.}}$, $\langle \frac{1}{N} \text{tr} B^\ell \rangle^{(1,0)}|_{\text{pert.}}$ and $\langle \frac{1}{N} \text{tr} W'(\phi)^\ell \rangle^{(1,0)}|_{\text{pert.}}$ by Z , $\langle \frac{1}{N} \text{tr} B^\ell \rangle$ and $\langle \frac{1}{N} \text{tr} W'(\phi)^\ell \rangle$, respectively. The Nicolai mapping

$$h_i = W'(\lambda_i) \quad \text{or} \quad H = W'(\Lambda), \tag{B.2}$$

recasts (B.1) as

$$Z_{(1,0)} = \tilde{C}_N \int d^{N^2} B \int_{-\mu^2}^{\infty} \left(\prod_{i=1}^N dh_i \right) \Delta(h)^2 e^{-N \text{tr} \left(\frac{1}{2} B^2 + i B H \right)}. \quad (\text{B.3})$$

From [3, 4], the effect of the lower bound of the integration region $[-\mu^2, \infty)$ with respect to h_i is considered to be nonperturbative in the $1/N$ expansion. We can replace the integrals by those over the whole real axis as far as the genus expansion is concerned. Thus the system we will consider is reduced to the standard Gaussian matrix model:

$$Z_{(1,0)}|_{\text{pert.}} = \int d^{N^2} B \int d^{N^2} H e^{-N \text{tr} \left(\frac{1}{2} B^2 + i B H \right)} = 1. \quad (\text{B.4})$$

The last equality follows from the normalization (2.6).

B.1 $\left\langle \frac{1}{N} \text{tr} B^\ell \right\rangle^{(1,0)}$

It is easy to see that the one-point functions

$$\left\langle \frac{1}{N} \text{tr} B^\ell \right\rangle^{(1,0)} \Big|_{\text{pert.}} = \frac{1}{Z_{(1,0)}|_{\text{pert.}}} \int d^{N^2} B \int d^{N^2} H \left(\frac{1}{N} \text{tr} B^\ell \right) e^{-N \text{tr} \left(\frac{1}{2} B^2 + i B H \right)} \quad (\text{B.5})$$

($\ell \in \mathbf{N}$) vanish due to the delta function with respect to the matrix B which arises from the H integral. Hence

$$\left\langle \frac{1}{N} \text{tr} B^\ell \right\rangle^{(1,0)} \Big|_{\text{pert.}} = 0 \quad (\text{B.6})$$

in all order in the $1/N$ -expansion.

B.2 $\left\langle \frac{1}{N} \text{tr} W'(\phi)^\ell \right\rangle^{(1,0)}$

Via the mapping (B.2), the one-point functions $\left\langle \frac{1}{N} \text{tr} W'(\phi)^\ell \right\rangle^{(1,0)} \Big|_{\text{pert.}}$ ($\ell \in \mathbf{N}$) becomes

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} W'(\phi)^\ell \right\rangle^{(1,0)} \Big|_{\text{pert.}} &= \frac{1}{Z_{(1,0)}|_{\text{pert.}}} \int d^{N^2} H \left(\frac{1}{N} \text{tr} H^\ell \right) e^{-N \text{tr} \frac{1}{2} H^2} \\ &= \tilde{C}_N \int \left(\prod_{i=1}^N dh_i \right) \Delta(h)^2 \left(\frac{1}{N} \sum_{i=1}^N h_i^\ell \right) e^{-N \sum_{i=1}^N h_i^2}. \end{aligned} \quad (\text{B.7})$$

We calculate this by using the orthogonal polynomials¹⁵. These are monic given by the Hermite polynomials:

$$P_n^{(H)}(x) = \frac{1}{(2N)^{n/2}} H_n \left(\sqrt{\frac{N}{2}} x \right) \quad (n \in \mathbf{N} \cup \{0\}) \quad (\text{B.8})$$

¹⁵Similar calculation is found in the correlation function of two “Wilson loops” in the one-matrix model in [27].

with

$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n H_n(-x). \quad (\text{B.9})$$

The orthogonality

$$\int_{-\infty}^{\infty} dx e^{-\frac{N}{2}x^2} P_n^{(H)}(x) P_m^{(H)}(x) = h_n^{(H)} \delta_{n,m}, \quad h_n^{(H)} = \sqrt{2\pi} \frac{n!}{N^{n+\frac{1}{2}}} \quad (\text{B.10})$$

is satisfied. By use of these properties and the fact that the constant \tilde{C}_N is expressed as $\tilde{C}_N = \left(N! \prod_{k=0}^{N-1} h_k^{(H)} \right)^{-1}$, (B.7) becomes

$$\left\langle \frac{1}{N} \text{tr} W'(\phi)^\ell \right\rangle_{\text{pert.}}^{(1,0)} = \frac{1}{N} \sum_{m=0}^{N-1} \frac{1}{h_m^{(H)}} \int_{-\infty}^{\infty} dx P_m^{(H)}(x)^2 x^\ell e^{-\frac{N}{2}x^2}. \quad (\text{B.11})$$

Clearly this vanishes for odd ℓ . Let us consider the case of even ℓ ($\ell = 2p$) in what follows. The orthogonal polynomials can also be expressed as

$$P_n^{(H)}(x) = N^{-\frac{n}{2}} \left. \partial_t^n e^{t\sqrt{N}x - \frac{t^2}{2}} \right|_{t=0} \quad (\text{B.12})$$

from properties of the Hermite polynomials. After plugging this into (B.11), straightforward calculation leads to

$$\left\langle \frac{1}{N} \text{tr} W'(\phi)^{2p} \right\rangle_{\text{pert.}}^{(1,0)} = N^{-p-1} (2p)! \sum_{r=0}^p \frac{1}{2^r r! ((p-r)!)^2} \sum_{m=p-r}^{N-1} \frac{m!}{(m-p+r)!}. \quad (\text{B.13})$$

Using the identity $\sum_{m=q}^n \binom{m}{q} = \binom{n+1}{q+1}$ and setting $n = p - r$, we finally obtain

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} W'(\phi)^{2p} \right\rangle_{\text{pert.}}^{(1,0)} &= \frac{(2p-1)!!}{N^p} F(1-N, -p, 2; 2) \\ &= \frac{(2p-1)!!}{N^p} \sum_{n=0}^{\infty} \frac{(1-N)_n (-p)_n 2^n}{(2)_n n!} \end{aligned} \quad (\text{B.14})$$

with $(x) \equiv x(x+1) \cdots (x+n-1)$ and $(x)_0 \equiv 1$. Hence the sum on n is actually a finite one. The first few results are explicitly given by

$$\begin{aligned} \left\langle \frac{1}{N} \text{tr} W'(\phi)^2 \right\rangle_{\text{pert.}}^{(1,0)} &= 1, & \left\langle \frac{1}{N} \text{tr} W'(\phi)^4 \right\rangle_{\text{pert.}}^{(1,0)} &= 2 + \frac{1}{N^2}, \\ \left\langle \frac{1}{N} \text{tr} W'(\phi)^6 \right\rangle_{\text{pert.}}^{(1,0)} &= 5 + \frac{10}{N^2}, & \left\langle \frac{1}{N} \text{tr} W'(\phi)^8 \right\rangle_{\text{pert.}}^{(1,0)} &= 14 + \frac{70}{N^2} + \frac{21}{N^4}. \end{aligned}$$

We can check that these are consistent with (4.28)-(4.31) for $W'(x) = x^2 - \mu^2$.

C Other derivation of instanton effects

In this appendix, we reproduce the instanton effect in section 5 from the viewpoint of distortion of the eigenvalue distribution by the instantons following the argument given in [27]. The partition function $Z_{(1,0)}$ can be written as

$$Z_{(1,0)} = \tilde{C}_N \int \left(\prod_{i=1}^N 2\lambda_i d\lambda_i \right) \Delta(\lambda^2)^2 e^{-N \sum_{i=1}^N \frac{1}{2}(\lambda_i^2 - \mu^2)^2} = \tilde{C}_N \int \left(\prod_{i=1}^N 2\lambda_i d\lambda_i \right) e^{-V_{\text{eff}}} \quad (\text{C.1})$$

with the effective potential

$$\begin{aligned} V_{\text{eff}} &\equiv N \sum_{i=1}^N \frac{1}{2}(\lambda_i^2 - \mu^2)^2 - \frac{1}{2} \sum_{i \neq j} \log(\lambda_i^2 - \lambda_j^2)^2 \\ &= N^2 \int dx \rho(x) \frac{1}{2}(x^2 - \mu^2)^2 - \frac{N^2}{2} \oint dx dy \rho(x) \rho(y) \log(x^2 - y^2)^2 \\ &\quad + C \left(\int dx \rho(x) - 1 \right). \end{aligned} \quad (\text{C.2})$$

$\rho(x) = \frac{1}{N} \text{tr} \delta(x - \phi) = \frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i)$ is the eigenvalue distribution, and C is a Lagrange multiplier imposing the constraint $\int dx \rho(x) = 1$. Similarly to the setting in the text, let us consider the p -instanton sector with $p = \mathcal{O}(N^0) \ll N$, where p eigenvalues are apart from the other $N - p$ eigenvalues. By relabeling the eigenvalues, it is natural to decompose $\rho(x)$ as

$$\rho(x) = \rho^{(0)}(x) + \frac{1}{N} \rho^{(1)}(x), \quad (\text{C.3})$$

$$\rho^{(0)}(x) = \frac{1}{N} \sum_{i=1}^{N-p} \delta(x - \lambda_i), \quad \rho^{(1)}(x) = \sum_{i=N-p+1}^N \delta(x - \lambda_i). \quad (\text{C.4})$$

Here $\rho^{(1)}(x)$ describes distribution of the isolated p eigenvalues. It follows from this definition that

$$\int dx \rho^{(0)}(x) = 1 - \frac{p}{N}, \quad \int dx \rho^{(1)}(x) = p. \quad (\text{C.5})$$

Substituting (C.3) for (C.2), we obtain

$$\begin{aligned} V_{\text{eff}} &= N^2 \int dx \rho^{(0)}(x) \frac{1}{2}(x^2 - \mu^2)^2 - \frac{N^2}{2} \oint dx dy \rho^{(0)}(x) \rho^{(0)}(y) \log(x^2 - y^2)^2 \\ &\quad + N \int dx \rho^{(1)}(x) \frac{1}{2}(x^2 - \mu^2)^2 - N \oint dx dy \rho^{(0)}(x) \rho^{(1)}(y) \log(x^2 - y^2)^2 \\ &\quad + C \left(\int dx \rho^{(0)}(x) + \frac{1}{N} \int dx \rho^{(1)}(x) - 1 \right) + \mathcal{O}(N^0). \end{aligned} \quad (\text{C.6})$$

The saddle point equation for $\rho^{(0)}(x)$ reads

$$0 = (x^2 - \mu^2)x - \oint dy \rho^{(0)}(y) \left(\frac{1}{x-y} + \frac{1}{x+y} \right) - \frac{1}{N} \oint dy \rho^{(1)}(y) \left(\frac{1}{x-y} + \frac{1}{x+y} \right) \quad (\text{C.7})$$

for x inside the support of $\rho^{(0)}(x)$. This equation implies that its solution also has the $1/N$ -expansion

$$\rho^{(0)}(x) = \rho^{(0,0)}(x) + \frac{1}{N} \rho^{(0,1)}(x) + \dots, \quad (\text{C.8})$$

where $\frac{1}{N} \rho^{(0,1)}(x)$ represents distortion of the eigenvalue distribution $\rho^{(0,0)}(x)$ in the large- N limit due to the presence of the p instantons. The solution to the equation (C.7) in the large- N limit (without the second term on the r.h.s.) has been already given in [10] as

$$\rho^{(0,0)}(x) = \frac{x}{\pi} \sqrt{(x^2 - a^2)(b^2 - x^2)} \quad \text{with} \quad a = \sqrt{\mu^2 - 2}, \quad b = \sqrt{\mu^2 + 2} \quad (\text{C.9})$$

for $x \in [a, b]$ ¹⁶. Plugging (C.8) into (C.7) and (C.5) provides conditions on $\rho^{(0,1)}(x)$:

$$0 = \oint dy \rho^{(0,1)}(y) \left(\frac{1}{x-y} + \frac{1}{x+y} \right) + \oint dy \rho^{(1)}(y) \left(\frac{1}{x-y} + \frac{1}{x+y} \right) \quad (\text{C.10})$$

and

$$\int dx \rho^{(0,1)}(x) = -p. \quad (\text{C.11})$$

C.1 $\rho^{(1)}(x)$

In order to find $\rho^{(1)}(x)$, we assume that the p eigenvalues are located at a saddle point $x = x_*$ outside the support of the perturbative configurations of a general filling fraction, i.e. $\Omega \equiv [-b, -a] \cup [a, b]$ and make an ansatz¹⁷

$$\rho^{(1)}(x) = p \delta(x - x_*). \quad (\text{C.12})$$

Then the effective potential in (C.6) becomes up to $\mathcal{O}(N)$

$$V_{\text{eff}} = (\text{\textit{x}}_*\text{-independent part}) + Np \left(\frac{1}{2} (x_*^2 - \mu^2)^2 - \oint dx \rho^{(0)}(x) \log(x^2 - x_*^2)^2 \right), \quad (\text{C.13})$$

¹⁶ For a general filling fraction (ν_+, ν_-) , it becomes

$$\rho^{(0,0)}(x) = \begin{cases} \frac{\nu_+}{\pi} x \sqrt{(x^2 - a^2)(b^2 - x^2)} & (x \in [a, b]) \\ \frac{\nu_-}{\pi} |x| \sqrt{(x^2 - a^2)(b^2 - x^2)} & (x \in [-b, -a]). \end{cases}$$

¹⁷ Since we are considering the case $p \ll N$, force between p eigenvalues can be neglected in this order.

whose saddle point equation $\partial_{x_*} V_{\text{eff}} = 0$ for x_* yields

$$0 = 2x_* \left(x_*^2 - \mu^2 - 2\text{Re} \langle R_2(x_*^2) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} \right). \quad (\text{C.14})$$

By noting (5.12) with (5.13), the solutions are

$$x_* = 0 \quad \text{or} \quad x_* \in \Omega, \quad (\text{C.15})$$

where the first one is appropriate to describe the position of the instantons. Thus

$$\rho^{(1)}(x) = p \delta(x). \quad (\text{C.16})$$

C.2 $\rho^{(0,1)}(x)$

Substituting back (C.16) for (C.10) leads to

$$0 = \oint dy \rho^{(0,1)}(y) \left(\frac{1}{x-y} + \frac{1}{x+y} \right) + \frac{2p}{x}. \quad (\text{C.17})$$

In order to solve this, we introduce a complex function

$$G(z) \equiv \int_a^b dy \frac{\rho^{(0,1)}(y)}{z-y}, \quad (\text{C.18})$$

and assume that $\rho^{(0,1)}(y)$ has the same support $[a, b]$ as that of $\rho^{(0,0)}(y)$. This assumption is plausible because the distortion $\rho^{(0,1)}$ is subleading in the $1/N$ -expansion of $\rho^{(0)}$ in (C.8) and the support will not move by $1/N$ corrections. Then (C.17) becomes

$$0 = G(x) - G(-x) + \frac{2p}{x} \quad \text{for} \quad x \in [a, b], \quad (\text{C.19})$$

and (C.11) leads to

$$G(z) \rightarrow -\frac{p}{z} \quad (z \rightarrow \infty). \quad (\text{C.20})$$

Therefore,

$$G_-(z) \equiv \frac{1}{2} (G(z) - G(-z)) = z \int_a^b dy \frac{\rho^{(0,1)}(y)}{z^2 - y^2} \quad (\text{C.21})$$

satisfies following conditions:

1. $G_-(z)$: odd, analytic in $z \in \mathbf{C} \setminus \Omega$.
2. $G_-(x) \in \mathbf{R}$ for $x \in \mathbf{R} \setminus \Omega$.
3. $G_-(z) \rightarrow -\frac{p}{z} + \mathcal{O}(1/z^3)$ as $z \rightarrow \infty$.
4. $G_-(x \pm i0) = -\frac{p}{x} \mp \frac{i\pi}{2} \rho^{(0,1)}(x)$ for $x \in [a, b]$.

From these conditions we can set

$$G_-(z) = -\frac{p}{z} + \frac{f(z)}{\sqrt{(z^2 - a^2)(z^2 - b^2)}} \quad (\text{C.22})$$

with $f(z)$ being odd. From the condition 3, $f(z) = \frac{\beta}{z} + \mathcal{O}(z^{-3})$ as $z \rightarrow \infty$. The analyticity at the origin in the condition 1 requires $f(z) = \frac{\beta}{z}$ with $\beta = -pab = -p\sqrt{\mu^4 - 4}$. Hence we arrive at

$$G_-(z) = -\frac{p}{z} - \frac{p\sqrt{\mu^4 - 4}}{z\sqrt{(z^2 - a^2)(z^2 - b^2)}}. \quad (\text{C.23})$$

Comparing this with the condition 4, we find the distortion

$$\rho^{(0,1)}(x) = \begin{cases} \frac{2}{\pi} \frac{p\sqrt{\mu^4 - 4}}{x\sqrt{(x^2 - a^2)(b^2 - x^2)}} & \text{for } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

C.3 Final result

Plugging the above results into

$$\langle R_2(z^2) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} = \int dy \frac{\rho(y)}{z^2 - y^2} \quad (\text{C.24})$$

with (C.3) and (C.8), we have

$$\begin{aligned} \langle R_2(z^2) \rangle_0^{(1,0)} \Big|_{p\text{-inst.}} &= \langle R_2(z^2) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} + \frac{1}{N} \left(\frac{1}{z} G_-(z) + \frac{p}{z^2} \right) + \mathcal{O}(N^{-2}) \\ &= \langle R_2(z^2) \rangle_0^{(1,0)} \Big|_{0\text{-inst.}} - \frac{1}{N} \frac{p\sqrt{\mu^4 - 4}}{z^2 \sqrt{(z^2 - a^2)(z^2 - b^2)}} + \mathcal{O}(N^{-2}). \end{aligned} \quad (\text{C.25})$$

It is easy to see that this is equivalent with (5.17).

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